ANNUAL REPORT

ON THE CHANGE OF FORM

HENRY BLUMBERG, Ohio State University

1. Strategy and mathematical exploration. Military science has its principles of strategy. So has chess,—as indeed, every game not of pure chance. The arts, too, have their basic principles of workmanship,—literature and oratory their "rhetoric," sculpture and painting their "techniques," *etc.* (We are not thinking of these principles in the derogatory sense of *mere* rhetoric or *mere* techniques, but in the broad sense comprehending their potential, intimate relationships with expression.)

How about mathematical exploration? Has it likewise an underlying strategy? As is inevitably the case for every developed field of human endeavor, the answer cannot but be in the affirmative.

And yet, while the strategy of war has from time immemorial engrossed the attention of military men, and has an immense literature devoted to it,—a somewhat similar assertion applying to literature and the arts—the "strategy of mathematical exploration" is likely to have a strange sound to mathematical ears. Is this not curious?

Every active mathematician is, of course, possessed of some system of motivating principles, which operate more or less consciously, clearly and dynamically. But mathematical literature has as yet not undertaken a broad and serious discussion of the "rhetoric" of mathematical exploration.

This phenomenon cannot be attributed to the deeper imaginative mysteries of mathematical invention as compared, say, with those of literature and the arts. Partly, perhaps, we may seek to explain it as due to the comparative recency of the grand development of mathematics, with its flood of new ideas in widely separated fields,—the call for strategic coordination being consequently delayed. In part, it may be due to the unique character of mathematical invention and discovery, which traditionally tends to regard every significant result as holding a peculiar secret of its own. Then there is the consideration that mathematicians, unlike artists and writers, say, do not much go in for sharp, open debate on the proper *modus operandi* for their vocation. (Scattered phases of how mathematicians work are treated in the autobiographical sketches of [6].)

2. What is "change of form"? But leaving it to the reader to assay the plausibility of these or other suggested explanations, we return to our main theme, which is to call attention to the need and potential utility of a more conscious and systematic strategy in mathematical work.

In the present article, we treat that phase of this strategy which we denote by "*change of form.*"

And what do we understand by "change of form"?

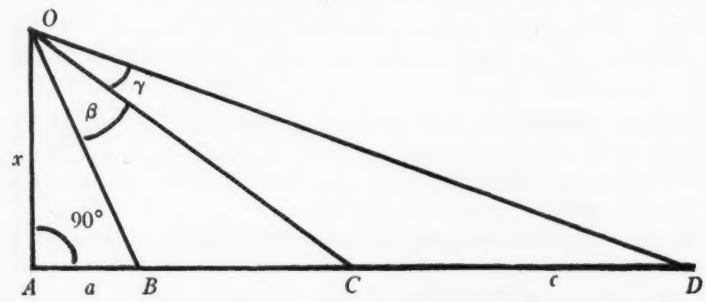
Military strategy involves the base of operations, the objective, and the line of operations. We might say—speaking in rough analogy—that change of form addresses itself to the line of operations. We could, without inexactness, use the

term *substitution* instead of *change of form*, but prefer the latter for its more appropriate abstractness and more dynamic connotation. For change of form combines both change and non-change—in a sense made clearer in the sequel. Or to put it otherwise, change of form is the "differential stroke" of mathematical exploration—in analogy to the similar role of the brush stroke in painting (*cf.* [1, 462–463]). And in accordance with this analogy, we may anticipate that the art of change of form, if duly developed, would be of basic service in mathematical exploration.

3. A preliminary remark. One more observation before proceeding with specificities. Principles of strategy are necessarily simple, and mostly obvious when pointed out. For example, the principle of a strong center in chess is anything but far-fetched. So is the principle which Rodin declared to be of prime importance in his work, namely, that of viewing every curve as the boundary of a surface, every surface as the boundary of a solid. But though all may nod quick assent to an announced principle, people differ very much in their grasp of its meaning and its range of application, and their ready use of it in situations where it can function. This consideration is of vital importance in the development of strategic power.

4. Types of change of form. We now turn to the classification and elucidation of the important types of change of form. Some types will appear as basic or primary, others as sub-types under these. Alternative classification may be devised with allowable transfer of our primary types to a secondary status, and vice versa. The virtue of a proposed classification lies not in its logical structure, but in its serviceability for exposition and exploration.

I. *Partial to comprehensive relationships.* An elementary illustration is furnished by the so-called "indirect problem" of freshman trigonometry. By this is meant a problem on triangles not directly reducible to the successive solution of triangles by the standard cases of trigonometry. For example:



Given $AB=a$, $CD=c$, $\angle BOC=\beta$, $\angle COD=\gamma$, $\angle A=90°$; find $AO=x$.

To solve this problem, we convert the slanted question of expressing x in terms of the given parts, into the more comprehensive one of determining a relationship among the five numbers a, c, β, γ and x. We are thus enabled to take

a, β, γ and x as known and c as unknown, rendering the problem direct (since we can compute c in terms of the other four letters by means of the standard cases of triangles, and thus secure the desired relationship connecting x and the given numbers).

By this method of treating knowns and unknowns on a par, all such indirect problems can be "solved"—*i.e.*, reduced to the solution of equations—whether the number of unknowns is one or more. It is frequently convenient to introduce additional variables.

The textbooks in trigonometry, as it seems, do not take up this general method, which removes an ostensible mystery.

Similarly, the so-called "reading problems" of elementary algebra, may be essentially reduced to computation. (Some recent algebra texts still speak as if a systematic plan for dealing with such problems were something not to be thought of.)

Another illustration is related to E. H. Moore's notion of "complete independence." For n postulates $P_1, P_2, \cdots, P_n$, we have 2^n possibilities with respect to the validity or non-validity of each. If there is an instance realizing each of these 2^n possibilities, the set of postulates is said to be "completely independent." If no instance exists for a particular one of these possibilities, we have a theorem—statable in one form or another—which asserts this non-realizability.

The situation is similar for n given properties (set properties, function properties, *etc.*), postulates, too, being interpretable as properties.

Curiously, a writer will sometimes take up just one or several of these possibilities, when there is no apparent reason why some of the other possibilities are of inferior interest or promise. This observation is not to be construed as an intended encouragement of problems of merely technical conception. When a certain type of work has lost its mystery for an artist—possessed of a high degree of faith in himself—he will turn to deeper mysteries. One versed in mathematical strategy will—*caeteris paribus*—more readily discern new problems, and consequently be more likely to bypass those of minor significance.

Under our present heading we may subsume the case of "putting the cart before the horse" or "reverse action." The applicability of such a maneuver may sometimes be overlooked by the student.

One instance of "reverse action" is the substitution of a variable for a number, as in the series

$$\frac{1}{1.2} - \frac{1}{2.3} + \frac{1}{3.4} - \frac{1}{4.5} + \cdots$$

for the purpose of evaluating it. Rewriting it in the form

$$\frac{1^2}{1.2} - \frac{1^2}{2.3} + \frac{1^4}{3.4} - \frac{1^6}{4.5} + \cdots$$

and substituting x for 1 in the powers 1^n, we have a power series which, upon two differentiations, yields a geometric series.

Similar substitutions of a variable for a number in the series $1-1+1-1+\cdots$ leads to its "evaluation," and thus suggests one way for introducing the definition of the sum of a divergent series.

Less familiar is the substitution of the general set for a point (interpreted as a set consisting of a single point), the general interval function (or property) for a particular interval function (or property), *etc.* (An interval function is an association of a number with every interval. An interval property α is a property pertaining to intervals such that every interval has or has not property α.) The role of such substitutions is illustrated under *Generalization* (*infra*).

Among other types that may be brought under the present heading, we mention the following:

a) *From expression to process.* An instance is the following generation of the polynomial $P(x)=a_0x^n+a_1x^{n-1}+\cdots+a_n$: Take a_0 and multiply by x, getting a_0x; add a_2 and multiply by x, getting $a_0x^2+a_1x$; add a_2 and multiply by x; *etc.* This reduction to process yields immediately a simple, graphical method for computing $P(x)$ for a given value of x, hence a graphical method for solving algebraic equations. (*Cf.* [5].)

b) *From symbol to meaning.* This type is illustrated in the generalization from positive integral, to fractional and negative exponents; from ordinary to fractional differentiation; from power series to "operational series"; and in general, from formal to actual entities (numbers, functions, *etc.*).

c) *Equivalences*, especially those revealing significant discriminations, as in the instances: axiom, postulate, assumption, definitional property; set, property, variable; abstraction, generalization, essence (*cf.* below under *Generalization*).

II. *Simplification.* We shall discuss this type under *Reduction*, *Decomposition*, and *Representation*, introducing subheadings under these.

1. *Reduction.*

a) *From the infinite to the finite.* This type is basic to the "arithmetization of analysis." Many graduate students, for example, may not quite grasp the salient motive for converting the equation $\lim a_n=a$ to a statement not involving the limit process—"for every positive ϵ there exists an n such that for every $\nu>n$ we have $|a_\nu-a|<\epsilon$. It would be good, in this regard, to bolster the student's sense of strategy by means of varied exercises exhibiting the gain through this change of form.

Similar remarks apply to alternative definitions of other concepts, such as continuity and *lim sup*, to arguments pertaining to double sequences, and to diverse other cases (*cf.*, for example, the discussion below under the change *from the exact to the approximate*).

In this connection, we may recall the retention by Kowalewski, in his *Grundzüge der Differential und Integralrechnung* [3], of the limit definition of continuity. This yields novel proofs, which, however, are not as simple and natural as when the ϵ, δ-definition is adopted, Moreover, the ready geometric interpretation of the latter speaks for its strategic superiority (see below under *Representation*).

b) *From the non-denumerable to the denumerable.* This type, analogous to a), ties in with the property of the continuum of containing a dense, denumerable set—a crucial property for proof and discovery. One consequence is the "replaceability" of all neighborhoods by a denumerable number of them, namely, the "rational" ones; another, the replaceability of a non-denumerable covering T (*i.e.*, a set consisting of a non-denumerable number of linear open intervals) by a denumerable subset of T; *etc.* No wonder, too, that this type of change of form comes naturally into evidence in the postulations for various abstract spaces.

c) To mention some other illustrations of reduction: In certain essential respects, a measurable set partakes of the character of (can be "reduced" to) the sum of a finite number of intervals; and a non-measurable set to a measurable one (*cf.* IV below). Likewise, a continuous function may be "reduced" to a step-function (*i.e.*, one composed of a finite number of intervals of constancy of the function); a measurable, to a continuous function; a non-measurable, to a measurable one (*cf.* [2, 276]). We should, however, underscore that such reduction is just "*in a certain sense.*" And it would be an interesting problem to secure a substantial clarification of this phrase, thereby rendering such "reduction" a more precise tool.

d) We add an observation in regard to the organization of high school geometry in the light of strategic principles—using the process of "reduction" as an illustration. Due elaboration of the principles underlying the art of reduction would greatly facilitate the discovery of proofs and of new theorems. But more interesting is the complementary, reverse process of "composition," which consists in building "compounds" from "elements." In elementary euclidean geometry, we would, for example, begin with a simple figure (like a circle and a point, or two circles), and "compose" a more involved one, thus gaining more opportunities for finding relationships, by adding "promising" construction lines, with whatever assistance strategic principles might afford. The variety of types of such constructions, even for the more complex "originals" that appear in the texts, is very restricted, as will be found upon examination. And simple principles for "promising" construction lines can be formulated without difficulty. (One such, for example, is that of the "*double-entendre*"—as we might call it—which enjoins the drawing of a new part that "fits," *i.e.*, has a simple relationship with, two parts already in the figure, and thus mediates a relationship that may be of some interest.) Thus, with a modicum of devoted interest and practice, every high school teacher could learn to compose "originals." And this is *a fortiori* so if the strategic principles are developed not merely for *reduction* and *composition*, but more comprehensively. Learning to do this is easier than, say, for an intelligent chess player to learn how to compose two-movers or three-movers.

Of course, many attempts have been made through the ages to organize, after one manner or another, a systematic approach to the solution of "originals"—not to speak of the discovery of new theorems. But, as it seems, these attempts were not actuated by a keen strategic sense, and a felt need of working

out the principles of mathematical strategy in its broad relations. *Compartmental* strategy, without the larger view, suffers from limitations that hinder success. Herein, as it appears to the writer, is to be found an explanation for the indifferent success hitherto attained in organizing the operative principles for proofs and "discoveries" in elementary geometry.

Similar remarks apply to the strengthening of the strategic power of students in projective geometry and other fields of geometry.

2. *Decomposition*, or breaking up into parts. This itself is a major type. Theoretically, it might be supposed, that consummate skill in decompositon could render exceptionally difficult problems accessible to solution by dint of a suitable gradation of steps.

To mention some illustrations: The fact that $\aleph_0 \cdot \aleph_0 = \aleph_0$ suggests decomposition in proving denumerability. Indeed, by means of this fact, and the one discussed above that the continuum contains a dense denumerable subset, we may without difficulty, prove virtually all the important theorems on denumerability appearing in treatises on point sets or real functions. And it would be advisable, for example, in graduate teaching, once the sense and art of strategy is developed, or to the end of promoting such development, to give as exercises the proofs of the denumerability of a variety of sets on the basis of the two facts just mentioned; such as that of the algebraic numbers, of the proper maxima of a real function, of the set of points which are limit points, from one side only, of a given linear set, *etc.*

I recall the incident of one of my beginning graduate students who seemed to be impressed, as barely any other student, with my remarks on the strategy of decomposition. Working on a theorem on denumerability (related to the fundamental problem of the calculus), he evolved a novel proof consisting of three successive decompositions (or was it four? I do not exactly remember). Most everybody would have lost heart before coming through this way, and have turned to other devices.

We mention additionally:

a) The trite example of breaking up the proof that $a=b$ into the cases $a \leqq b$, $a \geqq b$.

b) The breaking up of postulates or properties into component parts for purposes of generalization—as when the inequality $|f(x)-f(\xi)| < \epsilon$, in the definition of continuity, is split into two, $-\epsilon < f(x)-f(\xi)$, $\epsilon > f(x)-f(\xi)$, and one of them used in introducing the generalized notion of semi-continuity.

3. *Representation.* This type includes numerous cases that come immediately to mind—such as the various representations of real numbers (by means of Dedekind cuts, sequences, continued fractions, products, *etc.*) and functions (by means of Taylor, Fourier, Dirichlet series, *etc.*), maps of all sorts, concrete representations of abstract groups, fields, spaces and other abstract systems.

We add the following remarks:

a) Dedekind's definition of infinite set—as one that may be biuniquely mated with a proper subset of itself—is a classic example of the strategic ad-

vantage, wherever possible, of replacing negative by positive definitions. The former permit only abstract logical inference; the latter characterize structure, furnishing ready stimulus for intuitive surmises and deductions.

It was Liouville's discovery of a positive property of a real number satisfying no algebraic equation of degree $\leqq n$ which enabled him at once to define (for the first time) a class of real, transcendental numbers.

An analogous idea, stressed by G. Hessenberg, is that the *reductio ad absurdum* should be used most sparingly, inasmuch as it rests only on deductive logic, and proves only one proposition, while a direct proof deals with actualities and representations, and yields factual results all along the way.

A curious example of representation as tool was furnished by I. Schur in deriving his theorem for the commutativity of two linear homogeneous differential expressions [4]. The proof is based on the introduction of "ideal," *infinite* differential expressions.

b) That something as abstract as an unconditioned set admits a strategic representation is truly remarkable, but this is the substance of Zermelo's illustrious theorem that every set may be normally ordered. The late chess master Emanuel Lasker (who was also a fine mathematician), when he was first informed of this theorem incisively remarked, "It can't be true—it would be too powerful." Indeed, this theorem offers the only available means for proving various basic theorems in algebra and geometry, not to speak of set theory.

c) A notable instance is furnished by Hilbert in his grand project of guaranteeing against mathematical paradox, the fundamental means being the representation of every proof by means of certain elemental entities and processes.

d) We finally mention the "algebraization" of euclidean and projective geometry, among other geometries. It should, however, be observed that the representation of a geometry by means of an algebra cannot become a valuable strategic tool unless the algebra is, or is made into, a flexible instrument for making up and solving problems. The literature, at times, flatly ignores this strategic consideration.

III. *Generalization.* This topic is discussed at some length in [1]. We shall confine ourselves here to summary statements and some observations.

Generalization is applicable to concepts, theorems and processes. For concepts, we have the strategic principle that (1) *generalization*, (2) *abstraction, and* (3) *extraction of essence are equivalent procedures.* To explain: *Generalization* is used in the sense of extension of domain, as when we pass from integers to rationals or to reals; *abstraction*, in the sense of *substituting "partial" for "total" conditioning*, as in the passage from function in the pre-modern sense (involving expressibility), to the modern conception, which discards this property; and essence is understood, of course, not in an absolute, but in a relative sense. The determination of essence, however, is not a mere subjective matter, but based on mathematical experience and cultivated insight, as when the essence of functionality comes eventually to be recognized universally as mere correspondence.

The equivalence of generalization, abstraction and extraction of essence does

not, of course, signify identity—as may be said of all proper changes of form. When one of these procedures is to be applied, the fact of its equivalence to the other two is of potential assistance in suggesting cues for its realization.

Another procedure equivalent to these three is *extension of meaning* as when fractional exponent, or fractional differentiation is invested with meaning for the first time.

The appreciation of these equivalences will fortify the strategic skill of the student with reference to the important process of generalizing concepts.

Following is an instructive example of the "strategic" generalization of a theorem:

One proof of the non-denumerability of the continuum runs as follows: If $\{x_n\}$ is a given sequence of real numbers, there is a closed interval I, of length <1, not containing x_1; a closed subinterval I_2 of I, of length $<1/2$ not containing x_2; a closed subinterval I_3 of I_2, of length $<1/3$, not containing x_3; and so on. There is just one point x common to all the I_n, $n=1, 2, \cdots$, and this point is different from each x_n, $n=1, 2, \cdots$. We may secure a generalized theorem by retaining the framework of this argument and substituting for x_n (interpreted as a set consisting of a single point) the (linear) set S_n and then, if possible, determining the condition to be imposed upon each S_n so as to permit the definition of the intervals I_n, with their requisite character for effecting the proof. We can see immediately that the desired condition is that S_n be non-dense. We thus obtain the following generalized theorem: For every given sequence of non-dense, linear sets, there exists a point contained in none of them. (The generalization to euclidean n-space is immediate.)

Strategically speaking, this generalized theorem might well have been the original incentive for introducing the notion of "exhaustible" set (=set of first category, as it is known in the literature), namely, of a set which is the sum of $\aleph_0$ non-dense sets.

We observe that to be a good generalizer one must be a good changer of form, every change of form—whether in the definition of a concept, formulation of a process, or proof of a theorem—potentially offering new cues for generalization. Thus other proofs of the theorem we have just generalized lead to other interesting generalizations.

We remark further that the question of what constitutes "essence" deserves circumspect consideration. To promote a clearer and more comprehensive understanding of "essence" in its various bearings, it would be desirable, in graduate work, to take up more frequently and thoroughly than is the present vogue, questions like the following: What—from the point of view of proposing a definition—are the "essential" properties of a measurable set? Of a topological space? Of the sum of a divergent series? *etc.*

IV. *From the exact to the approximate.* We cite the following interesting illustration: Suppose we had the problem of developing in a graduate class a plausible generalization of the *length of an interval.* We might then take the following line of reasoning: Let us try to apply the change of form "from the exact to the

approximate." Knowing the meaning of length for an interval—hence for a finite interval sum (*i.e.*, the sum of a finite number of intervals)—we are to conceive of an appropriate approximation of a finite interval sum. A promising approximation would be a *refined* one—one differing from a finite interval sum by "as little as you please." A first thought, then, is to take as the desired approximation a (linear) set which, for every positive ϵ is representable in the form $T-E_1+E_2$, where T is a finite interval sum, and E_1 and E_2 are each coverable by intervals (finite or infinite in number) of length sum $<\epsilon$. We call such a set an "*approximate interval sum.*"

It turns out that this definition is equivalent to the definition of a measurable set, as first given by Lebesgue. Thus an *L-measurable set is nothing other than an approximate interval sum*, in the sense just defined.

The latter conception furnishes a representation, a structural picture, of a measurable set, in contrast to the descriptive definition of Lebesgue, and accordingly lends itself more readily to intuitive surmises of relationships and proofs. It offers, indeed, the simplest and most direct way of developing the basic theory of L-measurable sets.

Jordan content may be treated similarly; so may measurable function.

If we wish, the change from the exact to the approximate may be subsumed under *generalization.*

We finally mention the following types:

V. *Relativization*—the change from the absolute to the relative. This is exemplified not only in the Theory of Relativity, but in many other fields. As is patent, the potentialities of relativization are far from having been exploited. In projective geometry, there is the well-known relativization of euclidean theorems—based on the relativization of parallel, midpoint, circle, perpendicular, distance, *etc.*—which can well be made a generating principle for the main substance of the subject, leading readily to the cues and the proofs of most of the principal theorems and results in the treatises on projective geometry. In the field of real functions, we have relative closure, relative measure, relative convergence, *etc.*

VI. *Logical changes of form.* As, for instance, from a proposition to its contrapositive. It may be mentioned that with the use of the contrapositive, the Borel theorem might be presented at the beginning of real function theory as an exercise immediately implied by the fundamental theorem of the Dedekind theory of the irrational (that for every Dedekind cut, the lower section has a maximum, or the upper a minimum).

VII. *Integral to differential, and conversely.* This signifies a passage from "local" to "universal" (*im Kleinen* to *im Grossen*) properties, and conversely. This type is a basic motivating idea of Relativity, as when action at a distance is replaced by field theory (differential equations). The Borel and Vitali theorems may be interpreted as furnishing a passage from "local" to "universal" properties. And in the light of this change, students will more readily recognize the strategic significance of covering theorems.

5. Some additional observations. In the interest of further elucidation we subjoin the following observations.

a) We wish to reinforce the point, made in §3, that there is nothing mandatory in our adopted classification of changes of form, the guiding objective in our choice having been expository convenience and strategic serviceability. There is no absolute criterion for determining which changes of form are basic, no more than there is a compelling reason for taking a particular one of the categories Truth, Beauty, Virtue, Will, Love, *etc.*, as more basic than the others in the formulation of a philosophy of life. Thus, for example, generalization may be made to appear as a type of change of form, or conversely, change of form may be subsumed under generalization. Likewise more or fewer categories may be listed. There is no reason, too, why personal predilections are to be excluded; and more detailed classifications may commend themselves in some fields. The crux is the availability and sharpening of basic thought-tools.

b) To amplify somewhat another observation of the Introduction: No matter how elementary an item of strategy may appear in its formulation, it is not to be assumed that, even for the expert, it is a facile instrument in its manifold interpretations and applications. What counts is not mere logical endorsement of a principle, but making it a reliable constituent of one's thought and work. The comprehensive mastery of a strategic principle, it would seem—whether in mathematics or other fields—requires that it be brought into consciousness, and then—but this is rare, perhaps only a distant ideal, for abstract principles—made into the substance of our second nature, as a ready thought-tool in its various bearings. Only thenceforward could we without loss afford to permit its "submergence" beneath workaday consciousness, and draw on its full power in that organically comfortable state.

And,—as is apparent in the case of the arts and the competitive skills, like tennis, golf, *etc.*,—it is not to be assumed that a mathematician, even if truly great, has necessarily developed a particular elementary strategic principle to a superlative degree.

c) To urge the importance of consciously appreciated, readily applicable, strategic principles, does not, of course, imply a diminished esteem for the deeper mysteries of creative work, suggested by such figures of speech as "flights of the imagination," "converse with the Muses," *etc.* The quintessence of Beauty has not been captured by man, yet those who devotedly seek it gladly use all available means of advancing their comprehension of it, including sustained cultivation of technical knowledge and skill—as the life histories of artists, writers and musicians amply attest.

No doubt technique and strategy may be permitted to become the operative enemies of the creative spirit, just as science has been utilized for destructive ends. But properly employed, strategy is a true ally of the intuitive and imaginative faculties.

Indeed, as in the controversy over "free enterprise" *vs.* government planning, the question here is not one of just *pro* or *con*, but of difference in conception as

to the kind and degree of strategy which may be "legitimately espoused." No one would knowingly favor mystification which analysis can dissipate—as in the "reading problems" of algebra, the "originals" of geometry, the "indirect" problems of trigonometry, and many analogous instances on the so-called "higher levels." And in general, it is well to refrain from a dogmatism which would pre-determine—inevitably, with a cloud of vagueness—the type or degree of technique one may "legitimately" cultivate.

One who does not pursue art may marvel interminably at the mystery of a drawing that calls for only a modicum of technique. But when such technique is acquired, the search for beauty and mystery mounts to higher planes. So in mathematics. The development of strategic skill should mean the withdrawal of interest from the minor mysteries and the growth of interest in the deeper ones.

6. Relation to teaching. Lastly, we make some remarks on the bearing of our subject upon teaching.

Strategic power may be developed by all who are dependably devoted and assiduous in its pursuit. Nobody is marked at birth as permanently incapacitated from attaining it. What part or degree of mathematical talent depends on strategic attainments has never been reliably determined. In the matter of developing the capacity for "creative" and "effective" writing, distinguished publishers, editors and writers differ widely on the "relative efficacy" of training *vs.* "natural endowment." Likewise, in the case of mathematics, it is well to avoid dogmatism in reference to gauging such "relative efficacy." Teachers, accordingly, have no scientific or moral justification for passing drastic, definitive judgments upon some of their students in the matter of their mathematical potentialities, that "so and so is clearly not cut out for mathematics," and the like. This does not mean that individual differences are not great, nor that the requisite combination of devotion and discipline can be easily found. But every conscientious teacher will want to be safeguarded against destroying faith on misunderstood or false premises.

Strategy cuts across different fields, discerning essential sameness under marked diversity. It thus signifies the acquisition of *power* as distinguished from *passive, factual knowledge.* The possession of such power must be a primary goal of all true education. Paraphrasing E. Mach, we may say that education consists in "advancement in economy of thought."

We have already indicated earlier—in particular, in connection with the organization of euclidean geometry—the feasibility of projecting a significant strategy for "discovering" theorems and devising proofs. Nothing contributes so much to the understanding of the publications of others as self-initiation as a writer. By the same token, right teaching cannot afford to neglect the eliciting of the powers of invention and discovery. Interest and devotion being assumed, means are available for developing inventive capacity. To this end, strategy must be reckoned a substantial aid.

One of the outstanding shortcomings in our graduate mathematics is that the

capacity of the students for *judging significance* is inadequately developed. Mathematical literature, too, is not free enough in give-and-take critical evaluations of mathematical publications. And there is a paucity of serious consideration and spirited discussion of higher ideals in research—*practical* ideals, realizable in the near or the more distant future.

To be sure, for progress toward higher ideals, diverse means are to be called into play. But here is not the place to enter into their discussion. We might mention, however, that strategy is one of these means, giving faith and clearing the decks for progressively more arduous undertakings. One versed in strategy—other things being equal—is a better asker of questions, is better prepared to discern the more important ones. (Though A knows subject S better than B, the latter may more readily discern significant problems in S if he is an expert question-asker.)

No mathematician should be satisfied with being merely an improver or subjoiner of the basic discoveries of others. Something like the Picard-Landau-Schottky-Caratheodory-Bernays . . . "Ideenkreis" typifies a phase of mathematical activity that could felicitously be much abated in favor of more independent exploration. (And this instance is cited, of course, with all due respect to the eminence of these mathematicians, and without derogation of the professional interest and quality of the follow-up contributions in question.) In this regard, literature and the arts offer a hint and higher directive, inasmuch as all artists and men of letters worthy of the name endeavor to understand and interpret life, not mediately, but in an original way, proper to their own essential individuality and experience.

As mathematical strategy comes to be more widely discussed, and its potentialities developed and more prevalently utilized, it will begin to play its proper role as a basic aid in mathematical education and research.

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GRADUATE TRAINING IN STATISTICS*

W. G. COCHRAN, Iowa State College

1. Introduction. The emergence of statistics as a possible field of specialization for mathematicians is rather recent. There has, of course, been some kind of a profession of statistics for many years—the professional statistical societies are over 100 years old both in this country and in England. But until about 25 years ago statistics was regarded mainly as a useful tool in economics and politics and the deliberations in the professional societies reflected this point of view.

The beginnings of research and teaching in mathematical statistics were the results either of the insight and labors of individual enthusiasts or of demands from the users of statistical methods for better guidance. In London Karl Pearson was active and prolific in research, teaching and applications from 1893 until his retirement in 1933. An inspiring leader, he attracted into the field a number of able young men. In 1911, after some years as professor of applied mathematics, he was appointed head of a department of applied statistics, a post which he held jointly with a new professorship in eugenics. Upon his retirement the department of statistics was headed by his son, E. S. Pearson, while the chair of eugenics was also occupied by a noted mathematical statistician, R. A. Fisher.

In India Mahalanobis, professor of Physics at Calcutta, started a statistical laboratory about 1927 and carried this on at his own expense. The laboratory gradually brought together a group of young Indian mathematicians and has now become one of the outstanding centers of research in mathematical statistics in the world. These developments, it may be noted, were outside of departments of mathematics. A similar case in this country occurred at Columbia, where Hotelling, appointed professor of economics in 1931, is also a distinguished mathematical statistician. In consequence, the graduate lectures and research in mathematical statistics were at first conducted almost entirely in the department of economics.

About the same time, however, some mathematics departments began to offer work in mathematical statistics. At Michigan, for example, Carver became associate professor of mathematics and statistics in 1925 and has done much to promote the welfare of mathematical statistics in this country. Presumably as a result of his presence, Michigan now has three professors of mathematics who specialize in mathematical statistics. The late Dr. Rietz exercised a similar influence at the University of Iowa. To mention a few other dates, mathematical statisticians were appointed at Cambridge in 1931, Princeton in 1933, Wisconsin in 1937, California in 1938 and Iowa State in 1939. In several of these and similar cases, at least part of the impetus came from agricultural experiment stations which appreciated the services of a consulting statistician. At Cambridge, for instance, the Readership in statistics was established in the School of Agriculture, though some lectures are given by the Reader in the Mathematical Tripos.

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Thus, while statistics is at present represented in mathematics departments only to a minor extent, there are a number of institutions which offer a more or less complete graduate program in mathematical statistics.

Statistics has benefited tremendously from the influx of mathematicians of good caliber. Frequently the same statistical problem occurs in different fields of application. The applied statistician is likely to solve the problem purely in terms of the specific situation in which it arose, without concerning himself with other possible applications of his result. The mathematician goes to the essence of the problem and tackles this independently of the field of application. Consequently there has been built up a central fund of statistical theory, now fairly substantial, to which all applied fields may turn for new or improved techniques. Further benefit comes simply from the greater mathematical power available. Many statistical problems, whose solution was beyond the resources of most people in the field, yield readily to treatment by advanced mathematical methods. Consequently, it is important for the future progress of statistics that able mathematicians be attracted to specialize in that field.

2. Sources of employment. Before considering what constitutes suitable graduate training in statistics, it is relevant to examine briefly the job opportunities for statisticians and the types of work that these jobs involve. The "Description of the profession of statistics," issued recently by the National Roster of Scientific and Specialized Personnel, contains the following account of sources of employment.

> "Statisticians are employed in the various agencies of Federal, state and local government; in educational and research institutions, both public and private; in commercial, utility and manufacturing firms; in business, labor and other types of associations; in social agencies; in insurance companies; and in advertising and market research firms."

It is evident that there is a wide range of possible sources of employment. At the moment educational institutions probably rank as the chief single source for young mathematical statisticians. A tabulation of the present employment of 17 persons known to me who recently received advanced degrees in statistics shows that 11 have gone to colleges and universities, almost all of these to posts which combine teaching and research in mathematical statistics with the provision of consulting help to research groups on the campus. Of the remainder, 4 have gone into the federal service and 2 into industry. Opportunities in these three fields seem relatively plentiful for the near future.

3. Type of work. A concise definition of the functions of the statistician is extremely difficult to give. The definition given in the National Roster pamphlet is as successful as any and carries some authority, having been compiled in collaboration with the two leading statistical societies in this country.

> "The statistician uses inductive reasoning, based on the mathematics of probability, to develop and apply the most effective methods for collecting, tabulating and interpreting quantitative information."

According to this definition, two characteristics distinguish the statistician. First, he is a specialist in the techniques for collecting, tabulating and interpreting data; second, he applies inductive reasoning in these functions. To illustrate, the statistician may be asked to draw a sample of the sugar-beet farms in this country, in order to provide, at minimum cost, certain information about sugar-beet growing which is needed for policy-making. He may be asked to help in the planning of an experiment to test the effectiveness of DDT as a killer of undesirable insects. He is likely to be consulted on the best machines and procedure for some complex tabulation. He may specify the conclusions which can be drawn validly from the results of the DDT experiment.

The role of inductive reasoning is vital. In practically all cases where statistical methods are used, the data which are collected or tabulated or interpreted are of interest only insofar as they represent some larger group of data which statisticians call a population. In the sugar-beet example, the information which is wanted is that which applies to the whole group of sugar-beet growers: the sample is used only as a saver of time and money. The population in this case is real and definite, assuming that one can define a sugar-beet grower without ambiguity. Similarly, in the DDT experiment, the object is likely to be to predict what will happen in future uses of DDT under certain specified conditions. The experiment is thought of as a representative of a whole series of future trials. Although the population in this example is less tangible, it nevertheless governs the statistician's thinking when he is engaged in planning the experiment.

Thus the theory of statistics deals essentially with the properties of samples drawn in a specified way from one or more specified populations, and with the inferences that can be made about the populations from a knowledge of the samples. The collection of data is regarded by the statistician as the operation of drawing from a population a sample with certain desired properties. Similarly, the first step in the interpretation of data is to consider whether the data can be assumed to be drawn from some specified population. Moreover, these populations are characterized by *variability* in that the elements which comprise the population are different from one another. Without such variability, statistical problems are either trivial or non-existent.

Within the profession of statistics, there is a wide range in the amount of theoretical knowledge that is used in a man's work. At one extreme, some statisticians confine themselves, apart from some teaching, to research in the mathematical theory. At the other, there are statisticians who are essentially administrators of action programs and who use their basic theoretical training only in that it gives them, so to speak, a statistical point of view in their work.

These considerations suggest that a well-rounded graduate program must be flexible and should make available to the student the following features:

1. A thorough training in the theory of sampling and in any supporting mathematics necessary for that study.

2. Some knowledge, at a fairly concrete level, of the applications of statistical theory.

3. For students who may undertake consulting work, some basic knowledge of an applied field in which statistics is used.

We consider now the various parts of the training program in more detail.

4. Training in statistical theory. Most present graduate programs start with the assumption that the student has little or no undergraduate preparation in statistics. The students come either from mathematics or from some applied field such as economics or biology. The latter students usually possess some acquaintance with the applied phases of the subject, have had their enthusiasm aroused during an undergraduate course. The mathematicians, on the other hand, may have elected statistics simply from an interest in applied rather than pure mathematics and may have attended at most a course in probability. This lack of undergraduate preparation, which limits the amount that can be achieved in the graduate training, is, I believe, temporary, since there has been a growth in the undergraduate training in statistics concurrent with the growth in graduate training. Some time will elapse, however, before good undergraduate training is available generally.

The three basic courses in statistical theory are probability and distribution theory; statistical estimation; and the testing of statistical hypotheses. Probability appropriately comes first, since it enters into all statistical techniques. The course in probability opens with a statement of the laws of probability for discrete and for continuous variables and usually proceeds to its main topic, distribution theory. The typical problem in distribution theory may be stated as follows. Given that the random variates $x_1, x_2, \cdots, x_n$ follow a known law of probability, find the law of probability followed by the known functions $\phi_i(x_1, x_2, \cdots, x_n)$, $i=1, 2, \cdots, k \leqq n$. This type of problem arises repeatedly from applications, where the x_i represent the data which are to be collected or interpreted, while the functions ϕ_i represent the summary statistics (*e.g.*, means, totals, percentages) on which inferences about the population are based. Further, the amount and type of information which a statistic ϕ_i can supply about the population clearly depends on the frequency distribution of ϕ_i.

Several methods are available for solving distribution problems. Expressions for the desired frequency distributions can always be written down either as complex summations or as multiple integrals. The problem therefore reduces to one of performing the summations or integrations so as to obtain a reasonably simple answer. Alternatively, the method of moment-generating functions or characteristic functions provides a solution by a double application of the Fourier integral theorem. In addition, a distribution problem can sometimes be interpreted as the problem of finding the volume of some solid in multi-dimensional euclidean space and resolved by metrical methods in geometry.

Many distribution problems in statistical theory are unsolved in the sense that the integrations and summations cannot be expressed in simple terms. A rather common situation is that a solution is possible if the sample is very small, *e.g.* has, two, three or four observations, and that a limiting form of the solution is easy to find when the sample is very large. In the intermediate range the solu-

tion is lacking. Accordingly, important parts of the course on distribution theory are those dealing with various forms of the law of large numbers and with methods leading to approximate solutions of distribution problems. The course should also include an account of the properties of those frequency distributions which occur commonly in the theory, such as the binomial, Poisson, Gaussian, Chi-square, Student's t, Fisher's z, and so on.

The second basic course, statistical estimation, deals with the problem of estimating the properties of the population from a knowledge of those of the sample. Given the type of population and sample and the characteristics to be estimated, the theory of estimation seeks foolproof mathematical rules which automatically lead to the best method of estimation. For certain types of population, these rules have been found. In general, the problem is much more difficult; indeed it is not entirely clear how to define a "best" estimate. Nevertheless, substantial progress has been made along productive lines.

In many uses of statistical methods, the point at issue is whether the population is of a certain specified type, or whether two or more populations are the same in certain characteristics. For example, knowing the cost of buying and applying a certain amount of artificial fertilizer to a crop, one can calculate the minimum increase in yield which must result from the fertilizer in order to make its use profitable at a given selling price for the crop. In an agricultural experiment on this question, the data would comprise two samples, one of crop yields without applied fertilizer and one with applied fertilizer. In the interpretation of these data, the interesting question might be: does the mean yield in the latter population exceed that in the former population by at least the amount required for the application to be remunerative? This is an example of a statistical hypothesis about two populations. The *test* of the hypothesis is some calculation which enables the experimenter to decide whether the data are or are not consistent with the hypothesis. Such decisions are, of course, subject to error. The theory of testing hypotheses deals with an analysis of the various types of error and attempts to discover mathematical rules which lead to a decision with the minimum risk of error.

Thus the main body of theory teaches the student how to solve distribution problems, how to estimate properties of the population and how to test statistical hypotheses. In addition, many special problems in estimation or in testing hypotheses occur so frequently in applications that the solutions to these problems should be part of the basic knowledge of the student. Sometimes these special techniques are taught as illustrations of the theory: sometimes they are given in special courses. Examples are the method of least squares, a technique for estimation of which the fundamental theorems were established by Gauss; and multivariate analysis, which deals with the relations among groups of variables.

5. Supporting mathematics. The essential courses in mathematics are (1) advanced calculus, particularly multiple integration and expansions in series, both of which arise repeatedly in the solution of distribution problems, and (2) the

algebra of determinants and matrices and of linear and quadratic forms, which forms the mathematical groundwork for the method of least squares and for much of the development in multivariate analysis. Both are standard courses in almost any mathematics department which gives advanced lectures. With a good grasp of these topics, the student should be able to follow most of the statistical theory.

In addition, a number of other branches of mathematics are useful in that they contribute to particular parts of statistical theory, although they do not form the thread running through all parts. How the statistician rates these depends on his predilections. Personally, I would place on the "most useful" level a course in numerical methods including interpolation, numerical integration and some finite differences and a course on complex variable as far as contour integration. Numerical methods are useful because the results of theoretical investigations generally require the construction of tables before they can be handled by the non-mathematical applied worker. Complex variable is a helpful tool in dealing with distribution problems.

On the next grade of usefulness, one might mention a number of topics: combinatory analysis, useful for discrete distribution problems; vector analysis and n-dimensional geometry, again useful for distribution problems: point-set and measure theory, which is needed for following some of the more fundamental developments in probability; finite groups, which has recently found applications to the theory of the design of experiments: and calculus of variations, which is the basis of the general theory of testing hypotheses. This list above covers a fairly extensive range in advanced mathematics and includes some topics which may not be taught by the mathematics department. Such topics can be dealt with either by indicating a short course of reading to the student at the appropriate time or by arranging seminars.

Since it is to be anticipated that other branches of mathematics will provide fruitful applications to statistics in the future, the list of useful supplementary mathematical knowledge is not likely to be static.

6. Training in applied statistics. Two types of training may be offered. The first consists of courses in which the relevant body of statistical theory is applied to some subject-matter field, such as economics, psychology, engineering or biology or to some special type of activity such as the conduct of sample surveys, the design of experiments, or the setting-up of a quality-control program in an industry. Such courses are usually attended also by workers in the applied field who wish to obtain some grasp of the contributions of statistics to that field.

Further, the student may receive some experience in the independent application of his statistical knowledge by answering letters containing queries, by assisting in consulting work and by helping to construct or criticize the plans for research projects. These tasks, in which the student must himself select the appropriate statistical theory for a problem, or develop new theory if needed, often present considerable difficulty even to the student who has a good grasp of theory.

7. Training in an applied field. The object is to give the student a sufficient acquaintance with the field so that he can cooperate intelligently with workers in the field and can translate statistical problems arising there into theoretical counterparts which really correspond to the situation in which the problems arose. For the mathematician, in view of time limitations, such training tends to consist of some courses which deal with the fundamental theory in the subject-matter field and some which describe current research in the field.

There is some doubt amongst statisticians as to the amount of time which should be allotted to the work in applications or in the applied field at the expense of mathematics or basic theory. The answer probably depends on the student. With a student of high general intelligence and good mathematical ability who is likely to be successful in most things that he undertakes, the best procedure may be to concentrate on theory and mathematics on the grounds that youth is the time to learn mathematics and that, should applications later attract him, he can learn what is needed by his own efforts. With an average graduate student whose development is less predictable, it should be remembered that the abilities required for theoretical statistics are considerably different from those required for applied statistics. The student's training should therefore give him the opportunity to learn whether his preferences and his skill lie in theory or in applications.

8. Concluding remarks. The program of study described above, if supplemented by the production of a thesis, is intended to constitute an adequate Ph.D. training. Courses for a master's degree, while naturally less comprehensive, should, I believe, follow the same general plan.

The administration of such a program requires coordinated effort from a number of departments. Where a cooperative spirit is present, this coordination can be secured in several ways: it is perhaps too early to be sure which is the best way. At both Columbia and Iowa State, for instance, the programs are at present under the general direction of a committee recruited from the departments concerned, the courses being given in the appropriate departments. At London, as has been indicated, there is a separate department of statistics. With the latter arrangement, it is advisable to have all teachers of courses in statistics at least part-time members of the department of statistics.

In conclusion, I should like to re-emphasize that statistics depends primarily on mathematics and mathematicians for its future development. Anyone interested in the progress of statistics cannot but hope that able young mathematicians will continue to be attracted into this field, which offers a wide range of useful and stimulating applications. Such mathematicians need not be regarded as lost or strayed from the fold. For while statistics has as yet contributed relatively little to repay its debt to mathematics, it is to be expected that new points of view and new problems arising in mathematical statistics will in course of time enrich the body of mathematical knowledge itself.

PAIRS OF TRIANGLES INSCRIBED IN A CIRCLE

R. GOORMAGHTIGH, Bruges, Belgium

1. Introduction. The notion of *mean* triangle connected with a given triangle in the sense defined by Delens [1] plays an interesting part in some properties of the triangle. Particularly, Ramler [2] has considered pairs of triangles having a common mean, and use will here be made of the same notion in a study of *any* pair of triangles inscribed in the same circle; this leads to the generalization of some theorems given in Ramler's paper.

2. Mean triangle related to a given triangle. Using complex coordinates, let a_1, a_2, a_3 be the turns corresponding to the vertices of a triangle $A_1A_2A_3$ inscribed in the base circle having center O, while Ω is the unit point, and let us denote by

$$s_1 = a_1 + a_2 + a_3, \qquad s_2 = a_2a_3 + a_3a_1 + a_1a_2, \qquad s_3 = a_1a_2a_3$$

the symmetric functions of the a_i. The vertices of the mean triangle connected with $A_1A_2A_3$ are then $s_3^{1/3}$, $s_3^{1/3}\omega$, $s_3^{1/3}\omega^2$, where $1+\omega+\omega^2=0$.

The conjugate to a will be denoted by $\bar{a}$.

Van Horn [3] has given a construction of the mean triangle similar to the one contained in our solution of a question proposed by Boutin [4]; this last author has pointed out [5] the relations between the mean triangle and the isodynamic points.

A very simple construction is derived from the geometric meaning [6] of s_3: the equation* to the Simson line of a point t for the triangle $A_1A_2A_3$ being

$$tx - s_3\bar{x} = (t^3 + s_1t^2 - s_2t - s_3)/2t, \tag{1}$$

s_3 is the point for which the Simson line is parallel to the circumdiameter through Ω. Hence the following construction:

If XY is the circumdiameter parallel to the Simson line of any point Z, the vertices of the mean triangle are the points dividing in the ratio 1:2 the two arcs XZ and the two arcs YZ.

3. Angle associated with a pair of triangles. Let b_1, b_2, b_3 be the turns corresponding to the vertices of a second triangle $B_1B_2B_3$ inscribed in the base-circle, and denote by

$$\sigma_1 = b_1 + b_2 + b_3, \qquad \sigma_2 = b_2b_3 + b_3b_1 + b_1b_2, \qquad \sigma_3 = b_1b_2b_3$$

the symmetric functions of the b_i.

If A and B are the points s_3 and σ_3, *i.e.*, the points whose Simson lines in $A_1A_2A_3$ and $B_1B_2B_3$ respectively are parallel to the circumdiameter $O\Omega$, then the angle $\theta=(OA, OB)$ is given by

$$e^{i\theta} = \sigma_3/s_3;$$

* The equation shows that the vertices of the mean triangle have their Simson lines respectively parallel to the circumdiameters passing through these vertices; these Simson lines are the cusp-tangents to Steiner's tricusp, envelope of the Simson lines of $A_1A_2A_3$.

we will call it the *angle associated with $A_1A_2A_3$ and $B_1B_2B_3$.*

The equation (1) shows further that the Simson lines of any point on the circumcircle with respect to $A_1A_2A_3$ and $B_1B_2B_3$ form an angle ϕ such that

$$e^{2i\phi} = \sigma_3/s_3.$$

If A and B are the points having their Simson lines in $A_1A_2A_3$ and $B_1B_2B_3$ parallel to any given direction, the angle $\theta = (OA, OB)$ is a constant.

The sides of the mean triangle corresponding to $A_1A_2A_3$ form angles equal to $\theta/3$, $\theta/3+120°$, $\theta/3+240°$ with the sides of the mean triangle corresponding to $B_1B_2B_3$.

The Simson lines of any point on the circumcircle with respect to the triangles $A_1A_2A_3$ and $B_1B_2B_3$ form a constant angle equal to $\theta/2$.

When the triangles have a common mean, $\sigma_3 = s_3$ and $\theta = 0$.

4. Orthopoles. Murnaghan [7], Godeau [8] and Ramler [2] have given an interesting property* about the common orthopole of the sides of a triangle with respect to another triangle having the same mean: we will now generalize it.

Apply to $A_1A_2A_3$ a rotation about O, the rotation angle being θ and the new position of $A_1A_2A_3$ being $A_1'A_2'A_3'$; then the coordinates of the points A_i' are $a_i\sigma_3/s_3$.

But the orthopole of A_2A_3 with respect to the triangle $B_1B_2B_3$, *i.e.*, the intersection of the Simson lines of A_2 and A_3 in the triangle $B_1B_2B_3$, is

$$(b_1 + b_2 + b_3 + a_2 + a_3 + b_1b_2b_3/a_2a_3)/2 = (\sigma_1 + a_1\sigma_3/s_3 + a_2 + a_3)/2.$$

Hence the following theorem:

A rotation about O through angle θ sends $A_1A_2A_3$ into $A_1'A_2'A_3'$, and a rotation about O through angle $-\theta$ sends $B_1B_2B_3$ into $B_1'B_2'B_3'$. The orthopoles of the sides of $A_1A_2A_3$ with respect to $B_1B_2B_3$ are the midpoints of the segments joining the orthocenter H_b of $B_1B_2B_3$ to those of $A_1'A_2A_3$, $A_1A_2'A_3$, $A_1A_2A_3'$, and the orthopoles of the sides of $B_1B_2B_3$ with respect to $A_1A_2A_3$ are the midpoints of the segments joining the orthocenter H_a of $A_1A_2A_3$ to those of $B_1'B_2B_3$, $B_1B_2'B_3$, $B_1B_2B_3'$.

When $A_1A_2A_3$ and $B_1B_2B_3$ have a common mean, $\theta = 0$, and we find that, in this case, the midpoint of H_aH_b is the common orthopole of the sides of each of the triangles with respect to the other.

5. Common line of images. As the line of images [10] of a point on the circumcircle, with respect to $A_1A_2A_3$, passes through H_a and is parallel to the Simson line, the only possible common line of images for the triangles $A_1A_2A_3$ and $B_1B_2B_3$ is the straight line H_aH_b with the condition $\theta = 0$, a result given by Ramler [2]:

Two triangles $A_1A_2A_3$ and $B_1B_2B_3$ inscribed in a circle have a common line of images when they have a common mean; the common line of images joins the orthocenters H_a and H_b of the triangles.

* In fact, this property had been noted before in a somewhat different form by Neuberg [9].

According to a well known property, if the parallels to H_aH_b through A_1, A_2, A_3 and B_1, B_2, B_3 cut the circumcircle again at A_1'', A_2'', A_3'' and B_1'', B_2'', B_3'', then the perpendiculars dropped from A_1'', A_2'', A_3'' and B_1'', B_2'', B_3'' on A_2A_3, A_3A_1, A_1A_2 and B_2B_3, B_3B_1, B_1B_2 respectively meet at the point whose line of images is common to the given triangles.

6. Common Simson lines. The conditions for the coincidence of (1) and the equation

$$\tau x - \sigma_3\bar{x} = (\tau^3 + \sigma_1\tau^2 - \sigma_2\tau - \sigma_3)/2\tau$$

to the Simson line of a point τ with respect to $B_1B_2B_3$, are

$$t/\tau = s_3/\sigma_3,$$

$$t^2/s_3 + s_1t/s_3 - s_2/s_3 - 1/t = \tau^2/\sigma_3 + \sigma_1\tau/\sigma_3 - \sigma_2/\sigma_3 - 1/\tau.$$

The three possible values of t are therefore given by

$$(2) \qquad (s_3 - \sigma_3)t^3/s_3^2 + (s_1 - \sigma_1)t^2/s_3 - (\bar{s}_1 - \bar{\sigma}_1)t + (s_3 - \sigma_3)/\sigma_3 = 0.$$

There are three common Simson lines corresponding, in triangle $A_1A_2A_3$ to three points forming a triangle T_a, and in triangle $B_1B_2B_3$ to three points forming a triangle T_b; the triangle T_a is sent into triangle T_b by a rotation about O, through angle θ.

Let h_a and h_b be the coordinates of the orthocenters of T_a and T_b; then

$$h_a = t_1 + t_2 + t_3 = -(s_1 - \sigma_1)s_3/(s_3 - \sigma_3),$$

$$h_b = \tau_1 + \tau_2 + \tau_3 = -(s_1 - \sigma_1)\sigma_3/(s_3 - \sigma_3).$$

Hence

$$h_a - h_b = -(s_1 - \sigma_1).$$

The segment joining the orthocenters of T_a and T_b is equipollent to the segment joining those of $B_1B_2B_3$ and $A_1A_2A_3$.

As we further have

$$t_1t_2t_3/\tau_1\tau_2\tau_3 = s_3^3/\sigma_3^3,$$

the angle associated with the triangles T_a and T_b equals three times the angle associated with the triangles $A_1A_2A_3$ and $B_1B_2B_3$.

It may also be noted that, in (2), $t_1t_2t_3 = -s_3/\sigma_3$ and, as the Simson lines of t_1, t_2, t_3 are concurrent when $t_1t_2t_3 = s_3$, we may state the following:

The common Simson lines are concurrent when the given triangles have opposite means.

Equation (2) shows also that, if $A_1A_2A_3$ and $B_1B_2B_3$ have a common mean, there is only *one* common Simson line; it corresponds to the same point on the circumcircle as the common line of images.

7. An ellipse associated with a pair of triangles. We have already noted that the Simson lines of any point on the circumcircle with respect to $A_1A_2A_3$ and $B_1B_2B_3$ form an angle $\theta/2$. The locus of their intersection has the following parametric equation:

$$x = (\bar{s}_2 - \bar{\sigma}_2)/2(\bar{s}_3 - \bar{\sigma}_3) + t/2 - (\bar{s}_1 - \bar{\sigma}_1)/2t(\bar{s}_3 - \bar{\sigma}_3)$$

and is the ellipse having as center the point

$$(\bar{s}_2 - \bar{\sigma}_2)/2(\bar{s}_3 - \bar{\sigma}_3),$$

the lengths of the axes being

$$\left| 1 \pm (s_1 - \sigma_1)^{1/2}(\bar{s}_1 - \bar{\sigma}_1)^{1/2}/(s_3 - \sigma_3)^{1/2}(\bar{s}_3 - \bar{\sigma}_3)^{1/2} \right|.$$

The locus of the intersection of the Simson lines of a variable point on the circumcircle, with respect to the triangles $A_1A_2A_3$ and $B_1B_2B_3$, is an ellipse having as axes

$$\left| R \pm \frac{H_aH_b}{2 \sin \theta/2} \right|,$$

R being the radius of the circumcircle.

8. Feuerbach points of tangential triangles. The coordinates of the Feuerbach points of the tangential triangles of $A_1A_2A_3$ and $B_1B_2B_3$ are $x_a = s_2/s_1$ and $x_b = \sigma_2/\sigma_1$. But

$$x_a = \bar{s}_1 s_3/s_1, \qquad x_b = \bar{\sigma}_1\sigma_3/\sigma_1$$

and

$$\frac{x_b}{x_a} = \left(\frac{s_1}{\bar{s}_1} \Big/ \frac{\sigma_1}{\bar{\sigma}_1}\right)\frac{\sigma_3}{s_3}.$$

The angle formed by the circumdiameters passing through the Feuerbach points of the tangential triangles of $A_1A_2A_3$ and $B_1B_2B_3$ is equal to the difference between θ and twice the angle formed by the Euler lines of the given triangles.

9. Isogonal conjugates. We consider the isogonal conjugates M_a and M_b of the midpoint M of H_aH_b in the triangles $A_1A_2A_3$ and $B_1B_2B_3$. The coordinate β of the isogonal conjugate of a point α in the triangle $A_1A_2A_3$ is given by [11]

$$\beta(1 - \alpha\bar{\alpha}) = -\alpha + s_1 - s_2\bar{\alpha} + s_3\bar{\alpha}^2.$$

Hence the coordinates μ_a and μ_b of M_a and M_b are such that

$$2\mu_a(1 - \mu\bar{\mu}) = s_1 - \sigma_1 - s_2(\bar{s}_1 + \bar{\sigma}_1) + s_3(\bar{s}_1 + \bar{\sigma}_1)^2/2,$$
$$2\mu_b(1 - \mu\bar{\mu}) = \sigma_1 - s_1 - \sigma_2(\bar{s}_1 + \bar{\sigma}_1) + \sigma_3(\bar{s}_1 + \bar{\sigma}_1)^2/2.$$

If it is noted that $\bar{s}_1 = s_2/s_3$, and if ν is the coordinate of the mid-point N of M_aM_b, then

$$8\nu(1 - \mu\bar{\mu}) = (\bar{s}_1^2 - \bar{\sigma}_1^2)(\sigma_3 - s_3)$$

and

$$64\,\overline{ON}^2(1-\mu\bar{\mu})^2=(s_1-\sigma_1)(\bar{s}_1-\bar{\sigma}_1)(s_1+\sigma_1)(\bar{s}_1+\bar{\sigma}_1)(s_3-\sigma_3)(\bar{s}_3-\bar{\sigma}_3).$$

Hence the following theorem:

If ϖ is the power, with respect to the circumcircle, of the mid-point M of H_aH_b, and if N is the mid-point of the segment joining the isogonal conjugates of M in the triangles $A_1A_2A_3$ and $B_1B_2B_3$, then

$$ON=\frac{R}{2\varpi}H_aH_b\cdot OM\sin\frac{\theta}{2}.$$

This generalizes a theorem given by Ramler [2]:

The isogonal conjugates, as to two triangles having a common mean, of the mid-point of the segment joining their orthocenters, are symmetric with respect to the circumcenter.

10. Pairs of triangles having a common orthocenter. When H_b coincides with H_a, the triangles $A_1A_2A_3$ and $B_1B_2B_3$ are circumscribed to a conic having O and H_a as foci.

Then the triangles T_a and T_b (section 5) are equilateral, and the locus (section 6) of the intersections of the Simson lines corresponding to a moving point on the circumcircle, with respect to $A_1A_2A_3$ and $B_1B_2B_3$, is a circle equal to the common nine-point circle of those triangles.

11. Triangles symmetric with respect to the circumcenter. In that case, $\sigma_1=-s_1$, $\sigma_2=s_2$, $\sigma_3=-s_3$ and equation (2) becomes

$$t^3+s_1t^2-s_2t-s_3=0.$$

When the given triangles are symmetric with respect to the circumcenter, the common Simson lines pass through the circumcenter.

The ellipse in section 6 is then inscribed in $A_1A_2A_3$ and $B_1B_2B_3$ and tangent to the sides of the triangles at the images of the feet of the altitudes with respect to the mid-points of the corresponding sides. The properties of that ellipse are are well known [12]; in this case, $\theta=\pi$ and the axes are $|R\pm OH_a|$.

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DISCUSSIONS AND NOTES

EDITED BY MARIE J. WEISS, Sophie Newcomb College, New Orleans, 18, La.

The Department of Discussions and Notes is open to all forms of activity in college mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

Beginning with the August-September number of the MONTHLY the editor of this department will be Professor E. F. Beckenbach, University of California at Los Angeles, Los Angeles 24, California. Manuscripts should henceforth be sent to him.

TWO THEOREMS ON MULTI-DEGREE EQUALITIES*

A. GLODEN, Luxembourg

THEOREM 1. *The multi-degree equality*

$$a_1, a_2, \cdots, a_p \overset{n-3}{=} b_1, b_2, \cdots, b_p \dagger \tag{1}$$

with

$$t = \frac{2(\sum b_i^{n-1} - \sum a_i^{n-1})}{(n-1)(\sum b_i^{n-2} - \sum a_i^{n-2})},$$

implies

$$(2)\quad \begin{cases} a_1, \cdots, a_p, t-b_1, \cdots, t-b_p \overset{n}{=} b_1, \cdots, b_p, t-a_1, \cdots, t-a_p, & n \text{ even}, \\ a_1, \cdots, a_p, t-a_1, \cdots, t-a_p \overset{n}{=} b_1, \cdots, b_p, t-b_1, \cdots, t-b_p, & n \text{ odd}. \end{cases}$$

Proof. By virtue of (1), the relations (2) are satisfied, regardless of the values of t, for all integral exponents from 1 to $n-2$ inclusive. For the exponents $n-1$ and n, the two resulting equations are both satisfied by the value of t given above.

A numerical example. From $1+5 \neq 2+3$ and $1^2+5^2 \neq 2^2+3^2$, with $t=13$, one immediately obtains

$$1, 5, 8, 12 \overset{3}{=} 2, 3, 10, 11.$$

I have previously shown the following theorem.‡

* Note by the translator, H. L. Dorwart. In 1938, Professor Gloden published a 90 page monograph on the Tarry-Escott Problem (see The Tarry-Escott Problem by Dorwart and Brown, this MONTHLY, vol. 44, 1937, pp. 613–626) under the title *Sur les Égalités Multigrades* (Beffort, Luxembourg). In 1944, shortly after his escape from Germany where he had been held in exile for forty months, he published a revised and enlarged German edition whose title is *Mehrgradige Gleichungen* (Nordhoff, Groningen, Holland).

† The notation means that the sum of the numbers on the left equals the sum of the numbers on the right for each of the first $n-3$ integral powers of the numbers.

‡ Euclides, Madrid, July 1944, pp. 518–519.

THEOREM 2. *The multi-degree equality*

$$a_1, a_2, a_3 \overset{m}{=} b_1, b_2, b_3, \qquad m = 2, 4, \tag{3}$$

whose terms satisfy the additional relations

$$a_1 + a_2 - a_3 = 2(b_1 + b_2 - b_3), \qquad a_3 \neq a_1 + a_2, \qquad b_3 \neq b_1 + b_2,$$

implies the following multi-degree equality with five terms per member true for the first four odd degrees

$$5a_1 - 3a_2 + 3a_3,\ 7a_1 - a_2 + a_3,\ -3a_1 + 5a_2 + 3a_3,\ -a_1 + 7a_2 + a_3,$$

$$2(b_1 + b_2 + 3b_3) \overset{m}{=} a_1 + a_2 + 7a_3,\ 3a_1 + 3a_2 + 5a_3,\ 2(3b_1 - b_2 + b_3),$$
$$2(-b_1 + 3b_2 + b_3),\ 3(a_1 + a_2 - a_3) \qquad m = 1, 3, 5, 7.$$

To solve (3), I have now established the following formulas

$$[1, -4, -2],\ [2, 0, -3],\ [3, -1, 1] \overset{m}{=} [1, 1, 3],\ [2, -4, -1],\ [3, 0, -2],\ m = 2, 4,$$

where $[r, s, t]$ represents the quadratic form $ru^2 + suv + tv^2$.

A numerical example. For $u = 4$ and $v = 1$, we have

$$29, 45, (-2) \overset{2,4}{=} 15, 46, 23 \quad \text{with} \quad 29 + 45 - (-2) = 2(15 + 46 - 23).$$

Applying Theorem 2, we obtain (after dividing both members by 4)

$$1, 33, 39, 65, 71 \overset{1,3,5,7}{=} 11, 15, 53, 57, 73.$$

CONSTRUCTION OF THE IN-FEUERBACH POINT

S. M. KARMELKAR, Akola, Berar, India

CONSTRUCTION. *Let ABC be a triangle. Draw AP, the altitude to BC. Let D be the middle point of BC. Let I be the center of the inscribed circle. Draw DX perpendicular to BC. Join PI and produce it to meet DX in X. Join DI and produce it to meet AP in Y; join XY. Draw IL perpendicular to BC; draw LZ perpendicular to XY from L. Then Z is the in-Feuerbach point.*

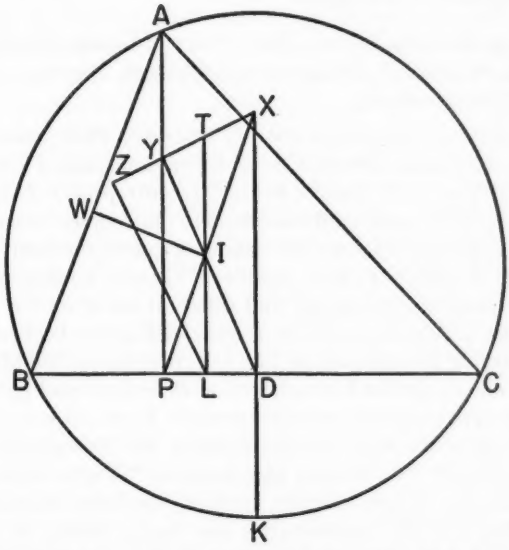
Proof. Use will be made of the following theorem: The angle formed by the altitude and the circumdiameter issued from the same vertex of a triangle is bisected by the bisector of the angle at the vertex considered. The angle equals half the difference of the base angles.*

Make the above construction and produce LI to meet XY in T. (See figure.) It is easy to prove the following properties of the construction: $PL/LD = YP/XD$; $IT = IL$; $\angle PYL = \angle LXD$; ZL bisects $\angle PZD$; the circle with TL as diameter passes through Z; ZI makes with ZL the same angle as ZL makes with the perpendicular from Z to PD, parallel to IL. The last statement and the above

* See Altschiller-Court, College Geometry, pp. 53, 59.

theorem show that ZI passes through the center of the circle PZD. The circle with center I and radius IL touches the circle PZD as they have the same tangent at Z, their common point.

It only remains to prove that the circle PZD is the nine-point circle. Draw the circle ABC. Draw DK, the perpendicular bisector of BC meeting the arc opposite to A in K. Join AK; I lies on AK. Draw IW perpendicular to AB.



Triangles AYI and IDK are similar. Hence $AY/DK=AI/IK$. Triangles AWI and CDK are similar and $CK=IK$. Hence $AI/CK=WI/DK=AI/IK$, and $AY=WI$, the radius of the inscribed circle. Hence AY is equal and parallel to IL; so $AYLI$ is a parallelogram. Hence $\angle PYL=\angle PAI$, and the latter equals $(\angle B-\angle C)/2$. Now PD subtends at Z an angle equal to double the angle which PL subtends at Z. Hence $\angle PZD=\angle B-\angle C$, and the circle PZD is the nine-point circle.

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

The Philosophy of Bertrand Russell (The Library of Living Philosophers, vol. V). Edited by P. A. Schilpp. Evanston and Chicago, Northwestern University, 1944. 16+815 pages. $4.00.

This is the fifth volume of *The Library of Living Philosophers*, the previous volumes dealing with John Dewey, George Santayana, Alfred North Whitehead, and G. E. Moore. The basic thought behind the enterprise is to help remove one of the greatest obstacles to fruitful discussion in philosophy, namely "the curious etiquette which apparently taboos the asking of questions about a philosopher's meaning while he is alive" (F. S. C. Schiller). There is no denying the fact that different contemporary interpreters find different ideas in the writings of the same philosopher. Could not at least some clarification be brought about by putting these varying interpretations and critiques before the philosopher while he is still in our midst, and let him act both as defendant and judge? That is the way in which the articles in this volume approach Bertrand Russell's philosophy. It is opened by one of the most charming essays Mr. Russell ever wrote, a brief autobiography entitled "My Mental Development." There follow 21 descriptive and critical articles on R's philosophy, covering the same extremely wide range as the protagonist's work: mathematics and logic, theory of knowledge, the philosophical aspects of science, of language and history; social, political and educational philosophy; psychology, metaphysics, ethics and religion. In 61 pages Russell boldly faces and responds to this not always too consonant chorus of philosophical voices, answering their questions, meeting a fair part of their many criticisms, and rectifying misinterpretations. A carefully compiled bibliography of his publications (45 pages) greatly increases the usefulness of the book for those who are anxious to study Russell's philosophy at the source. Also the detailed index deserves a word of praise.

The articles—how could it be else?—vary greatly in tone and value. Some, as, *e.g.*, Reichenbach's on Logic, are content to outline R's essential ideas in the field under discussion and ask a few relevant questions. Others attempt a detailed critical evaluation. Such, as one would expect, is Gödel's article on Mathematical Logic. An example of the best one can hope to accomplish in a philosophical dispute is the exchange of views on Philosophy of Science between E. Nagel and R. Here the disputants, critical but fair-minded, meet as it were on the same level. Einstein's criticisms are less detailed but sustained by his profound experience in the actual creation and operation of physical theories,—an experience which Russell lacks. Einstein is a great admirer of Russell's work, to the reading

of which, as he confesses, he owes innumerable happy hours. Let us pass in silence other articles where the critics show themselves clearly incapable of doing justice to R's thoughts or are more anxious to ride their own hobby-horse into the arena than to face the issues raised by R's philosophy. In spite of such shortcomings, inevitable in a community where freedom of thought is treasured, the book proves that philosophical discussion can benefit greatly by concentration on the work of one philosopher, especially if that philosopher can still be asked for his own interpretation and is as versatile and frank as Russell in his answers. The book, however, does not provide, nor does it intend to provide, a substitute for first-hand contact with the original thought of the philosopher himself.

Great is the temptation for the reviewer to quote profusely from Russell's introduction which, in sketching his own life and intellectual adventures, conjures up a whole period. Speaking of his friends at Cambridge in the nineties and their activities he exclaims: "For those who have been young since 1914 it must be difficult to imagine the happiness of those days." Here is a little illuminating story: "I remember the precise moment, one day in 1894," so R tells us, "as I was walking along Trinity Lane, when I saw in a flash (or thought I saw) that the ontological argument is valid. I had gone out to buy a tin of tobacco; on my way back, I suddenly threw it up in the air and exclaimed, as I caught it: Great Scott, the ontological argument is sound." Can anyone, after this anecdote, doubt that Russell is constitutionally a philosopher and not a mathematician?

In his "Reply to Criticisms" Russell draws a sharp line between that part of philosophy which is concerned with facts, and that which depends on ethical considerations or value judgments. In his opinion ethical propositions should be expressed in the optative mood, not in the indicative. They express desires rather than facts, although he agrees with Kant that beyond expressing a desire "ethical judgments must have an element of universality. I should interpret '*A* is good' as 'Would that all men desired *A*'." For this reason, he continues, "I do not offer the same kind of defence for what I have said about values as I do for what I have said on logical or scientific questions." In his replies touching on social, ethical and educational matters Russell occasionally uses the weapon of irony, though more sparingly than one would perhaps expect. Even when he begins with a sentence like this: "Mr. Brightman's essay on my philosophy of religion is a model of truly Christian forbearance," he continues with a dispassionate and frank restatement of his religious attitude. In only one instance, Bode's paper on Educational Philosophy, he shoots back with sharp arrows.

From among the articles a few more which concern the theory of scientific knowledge ought to be cited, at least by title: G. E. Moore, to whom Russell in his youth owed his emancipation from Hegelianism, writes about R's Theory of Description, J. Feibleman on the Introduction to the Second Edition of the Principles of Mathematics, P. P. Wiener on the Method of R's (early) work on Leibnitz, A. P. Ushenko on his Critique of Empiricism, and R. M. Chisholm on the Foundations of Empirical Knowledge. In the face of such an *embarras de*

richesse the reviewer plans to focus his attention on the two articles of the most undeniable interest to mathematicians, Reichenbach's and Gödel's essays on Logic, and on E. Nagel's discussion of R's Philosophy of Science. If *Principia Mathematica* can be called a critique of naïve set-theory, then Gödel's article is a critique of this critique, and what I, as the reviewer, am expected to write now, is a critique of a critique of a critique. This makes my task so difficult that I see no other way out than to base my review on a brief independent survey of the situation; it adds one more feature to the picture: the historical perspective.*

In the essay on Logic Reichenbach gives Russell his due credit for improving the formal apparatus of symbolic logic, in particular for the consistent use of propositional functions with their variables, and for his interpretation of the "If then" as material implication $\rightarrow$. Indeed, it is used in this sense throughout mathematics, and only material implication is a primitive logical operator like $\sim$, $\cap$, $\cup$. However, Reichenbach's account of the ramified theory of types misses the essential point. In all questions concerning the logical foundations of mathematics he has the tendency to oversimplify the issues. His remarks on pp. 43–44 turn Brouwer's wine into water. Neither Reichenbach nor Nagel challenges the Frege-Russell thesis that logic and pure mathematics are essentially identical and that the theory of natural numbers with its definitions and inferences by complete induction, is reducible to logic. The Survey shows why I do not believe in this thesis at all; I find, on the contrary, that since Burali-Forti's and Cantor's discovery of the antinomies, the whole drift of research has been away from it.

In his reply Russell argues for the law of excluded middle, and against the definition of truth in terms of verifiability. But he argues on a level that has little to do with Brouwer's deep insight into the fallacy of that law with regard to existential propositions. Russell wishes to keep room for such unverifiable truths as he feels are "necessary for the interpretation of beliefs which none of us, if we are sincere, are prepared to abandon." The following dictum of his should be taken to heart: "As logic improves, less and less can be proved." Reichenbach ventures beyond deductive logic by opening a discussion with Russell on the principles of inductive logic on which the inductive methods of natural science depend; but they find little common ground.

Kurt Gödel's paper on R's Mathematical Logic is the work of a pointillist: a delicate pattern of partly disconnected, partly interrelated, critical remarks and suggestions. It must have been very difficult to arrange them in the one-dimensional sequence to which the writer, unlike the painter, is forced by his medium of words. With caution and circumspection the argument touches on a great variety of possibilities (and pitfalls) of interpretation. Russell varies the formulation of his vicious circle principle as follows: "No totality can contain members *definable* only in terms of this totality, or members *involving* or *presupposing* this totality." He himself obviously thinks that it makes little differ-

* Mathematics and Logic, this MONTHLY, January 1946 [quoted as *Survey*].

ence whether one says "definable" or "involving" or "presupposing." But Gödel sees here three different principles, and the acceptance or rejection of one or the other depends above all on whether the members are *given* or *constructed*. I think, with the latter contrast he touches on a very significant point. As far as the Survey preceding this review elaborates the same point, it is nothing but a comment on Gödel's paper. The constructive definition of the lowest level in the hierarchy of levels, first given in my book *Das Kontinuum* some thirty years ago, is in full accord with this view. About Russell's axiom of reducibility Gödel says that "it is demonstrably false unless one assumes either the existence of classes or of infinitely many 'qualitates occultae'."

In addition to Russell's own work concerning the solution of the paradoxes, two other possible directions are mentioned in which, as Russell himself had pointed out, a "criterion for the existence of classes" might be sought, namely the theory of limitation of size exemplified by the Zermelo-Fraenkel axioms, and a zigzag theory toward which Gödel sees Quine's recent system moving.* After disposing of Ramsay's escape by means of propositions of infinite length, he envisages as a further possibility, though somewhat vaguely, a theory in which "every concept is significant everywhere except for certain 'singular points' . . . so that the paradoxes would appear as something analogous to dividing by zero. . . . Our logical intuitions would then remain correct up to certain minor corrections, *i.e.*, they could then be considered to give an essentially correct, only somewhat 'blurred' picture of the real state of affairs".

In order to appreciate his last sentence one must know that Gödel takes the paradoxes very seriously; they reveal to him "the amazing fact that our logical intuitions are self-contradictory." This attitude toward the paradoxes is of course at complete variance with the view of Brouwer who blames the paradoxes not on some transcendental logical intuition which deceives us, but on a gross error inadvertently committed in the passage from finite to infinite sets. I confess that in this respect I remain steadfastly on the side of Brouwer.

On the ground of all his experience Gödel makes a strong plea for the realistic standpoint where classes are conceived as real objects, namely as "pluralities of things," or as structures consisting of such pluralities, and he adds: "It seems to me that the assumption of such objects [classes or concepts] is quite as legitimate as the assumption of physical bodies and there is quite as much reason to believe in their existence. They are in the same sense necessary to obtain a satisfactory system of mathematics as physical bodies are necessary for a satisfactory theory of our sense perceptions and in both cases it is impossible to interpret the propositions one wants to assert about these entities as propositions about the 'data.' Logic and mathematics (just as physics) are built upon axioms with a real content which cannot be 'explained away'." In his opinion the two major attempts in this realistic direction are "the simple theory of types (in an appropriate interpretation) and axiomatic set theory." He adds the warning: "Many symptoms show only too clearly however that the primitive concepts need further clarification."

* This MONTHLY, vol. 44, 1937, pp. 70–80.

Gödel hardly mentions Brouwer and Hilbert, and the last remarks sound as if he were prepared to retire from their advanced positions to some axiomatic setup of the Zermelo type. Does this mean that he has given up all hope of settling things along the lines of Hilbert's approach, and is content to adjourn the problem of consistency *ad calendas Graecas*? It is impossible to discuss realism in logic without drawing in the empirical sciences. In defending his program against Brouwer, Hilbert pointed emphatically to the situation in theoretical physics. The individual physical statements and laws have no "meaning" verifiable in immediate observation; only the system as a whole can be confronted with experience. Here consistency is absorbed into the farther-reaching requirement of "*concordance.*" Every indirect determination of the value of a physical quantity in a concrete case (*e.g.*, the charge of an electron) is based on theories establishing functional relations between that quantity and others amenable to direct observation (as, *e.g.*, position of a pointer on a scale). Concordance means that, with due regard to the inaccuracy of measurements, all such determinations must lead to the same value. We never can be sure whether such concordance, however complete for the moment, will still survive when our observations expand and become more accurate. Why then should we wish that consistency, the mathematical part of concordance, be assured *a priori* for all future? Is it not enough that the system has met the test of all our elaborate mathematical experiments so far? Will it not be early enough to change the foundations when, at a later stage, discrepancies appear? That is a position against which I cannot find much to say. But what shall be the guiding principles of our theoretical constructions? Gödel, with his basic trust in transcendental logic, likes to think that our logical optics is only slightly out of focus and hopes that after some minor correction of it we shall see *sharp*, and then everybody will agree that we see *right*. But he who does not share this trust will be disturbed by the high degree of arbitrariness involved in a system like Z, or even in Hilbert's system. What is the supporting faith? Success alone cannot be the answer. How much more convincing and closer to facts are the heuristic arguments and the subsequent systematic constructions in Einstein's general relativity theory, or the Heisenberg-Schrödinger quantum mechanics! A truly realistic mathematics should be conceived, in line with physics, as a branch of the theoretical construction of the one real world, and should adopt the same sober and cautious attitude toward hypothetic extensions of its foundations as physics exhibits. Theoretical physics today is in a healthier state than mathematics. But neither Gödel nor I can offer concrete suggestions for a cure, although Gödel seems to hope for clues from a careful study of Leibnitz's notes on his project of a *characteristica universalis*.

By talking about physics we have already passed into the domain of Nagel's controversy with Russell on Philosophy of Science. Unfamiliar with R's pertinent systematic publications, I must limit my discussion to this controversy as it reflects itself in my own mind. The following is typical for the pointed arguments by which Russell sometimes likes to reduce an intricate epistemological

problem to a simple formula: "Common sense says: 'I see a brown table.' It will agree to both the statements: 'I see a table' and 'I see something brown.' Since, according to physics, tables have no color, we must either (a) deny physics, or (b) deny that I see a table, or (c) deny that I see something brown. It is a painful choice; I have chosen (b), but (a) or (c) would lead to at least equal paradoxes." In looking upon things as "bundles of qualities" and letting physics tell us that tables have no color, Russell remains pretty close to the favored epistemological slogan of the 19th century that only the raw material of sensations is immediately given to us, and to the doctrine, as old as Democritus and Huyghens, that nothing but "the atoms and their motions exist in reality" or that light and color are "in reality" oscillations of the ether. But in the development of physics the physical concepts have revealed themselves more and more as free constructions, mere *symbols* handled according to certain rules; theoretical physics becomes a system as thoroughly formalized as Hilbert's mathematics. Concerning the other side, the "given," many philosophers are now willing to agree that the sensory data, far from being given in their naked purity, are theoretical abstractions, while the true raw material is the manifest disclosed world in which man finds himself in his everyday life. It is quite characteristic that Hilbert bases his mathematics on the practical manipulation of concrete symbols rather than on some "pure consciousness" and its data. Thus quantum mechanics sets the physical phenomena symbolized by the Schrödinger-Dirac ψ's against the actual measurements for whose description our common sense "workshop" language is sufficient and adequate.

"Occam's razor," the search for a "minimum vocabulary," are some of the favored terms in which Russell describes the gist of his method. He pronounces as his supreme maxim of scientific philosophizing: "When possible substitute constructions out of known entities for inferences to unknown entities." Russell's goal is "to find an interpretation of physics which gives a due place to perceptions." Nagel criticizes in detail the steps by which he arrives (allegedly) at the twofold conclusion that (1) all the objects of common sense and developed science are logical constructions out of *events*, and that (2) our perceptions are events, but there exist many events which are not perceptions. I am inclined to side with Nagel, in particular when he says (p. 345, footnote) with regard to R's definition of point-instances in terms of events (dubious even from a purely mathematical standpoint): "One need only compare [this definition] with such analyses as those of Mach concerning mass and temperature, or those of N. R. Campbell concerning physical measurements to appreciate the difference between an analysis which is quasi-mathematics and an analysis which is directed toward actual usage." When Russell speaks of perceptions as events he does not seem to use the term in its "immanent" sense for which it is true that an actual perception is always *my* perception.

The physicist of today sees no need to use as material of his constructions some such ultimate entities as events; the pure symbols are enough for him. But he can handle these symbols and carry out his measurements only as a man

living in and accepting our common sense world. Of course the connection between theory and observation must be understood, and it is the main task of an epistemological analysis to make this understanding explicit. Nagel comes to a similar conclusion: "It seems to me, therefore, that, instead of making the elimination of symbols for constructs the goal of the logical analysis of physics, a more reasonable and fruitful objective would be the following: to render explicit the pattern of interconnections between constructs and observations, on the strength of which these latter can function as relevant evidence for theories about the former." With this sober statement I think both Gödel and Einstein would agree.

But the debate on questions of this sort will probably go on forever. The volume under review is likely to stir up a host of thoughts in the mind of every reader. Even a mathematician must at times grapple with the problem to understand "what it is all about."

HERMANN WEYL

Analytic Geometry. By R. R. Middlemiss. New York, The McGraw-Hill Book Company, 1945. 7+306 pages. $2.75.

The author has presented a well organized text in analytic geometry which is directed toward the needs and interests of not only the future major in mathematics, but also the engineer, chemist, or physicist. Excellent taste has been shown in the selection of the topics—especially those which forsehadow the requirements of calculus. For example, as the author points out in the preface, the amount of emphasis upon formal study of conic sections has been minimized to those essentials which may be useful to the technical student. Moreover, greater emphasis has resulted in the treatment of algebraic functions including a well chosen chapter upon polynomial curves, a chapter upon rational fractional functions, and a chapter upon other algebraic curves. Since the student of the calculus encounters the circle, ellipse, or hyperbola less frequently than he does other algebraic functions, this shift of emphasis is fortunate. Also greater attention is given to the treatment of transcendental functions and curves (including a section on hyperbolic functions) which are more common than the conic sections in calculus. As a result, the student is able to grasp better the actual properties of mathematical functions, the concepts of which are introduced in the first chapter and fortunately repeated throughout the text.

Dr. Middlemiss has included many other noteworthy principles which too frequently have been neglected in a student's preparation for the calculus. He has wisely emphasized addition and multiplication of ordinates as a means of ascertaining through a quick sketch the general graphical representation of a mathematics function, another valuable forerunner to the student's needs of calculus. Another feature of interest to the future calculus student is the introduction of the limit concept in a section concerned with horizontal asymptotes and the presentation of a few elementary problems in mechanics including the concept of moment of force.

Aside from the topics mentioned, polar coordinates and parametric equations are included in separate chapters. Of interest to any person in experimental work is an adequate chapter about curve fitting. Such a chapter will be valuable to the research worker in biological sciences, psychology, education, and statistics as well as to the student in the physical sciences. The last section of the book is devoted to solid analytic geometry. In the three chapters upon solid analytics, the author appears to reach his greatest clarity in exposition. The excellent figures which represent three dimensional properties are extremely helpful to the student who may have trouble in visualizing the geometry of space. In fact, all the figures throughout the text are noteworthy for their aid in explanation and for their ease of interpretation. At the end of the text are tables of logarithms (four place) of trigonometric functions (both natural and logarithmic values), of powers and roots, and of natural logarithms. The author should be complimented for giving answers to nearly all problems, both "even" and "odd" numbered examples.

Other meritorious features of the text are numerous. An introductory review chapter of the essential concepts of algebra and trigonometry required in analytics is helpful. The explanation of directed line segments, projection, and point of division is particularly good. Other topics which are especially well presented to the student include the normal form, systems of lines, and algebraic curves.

The only criticism that this review can offer is that in certain sections of the text a greater number of illustrative examples would be helpful. The explanation does seem somewhat brief with respect to translation and rotation—especially in the way of illustrative problems. Concerning the ellipse no illustrative example is presented other than that for finding certain properties of an ellipse given its equations. A few illustrative problems for the parabola, ellipse, and hyperbola in which properties of the curve are given in order to find the equation seem to be very much needed. Although the author is wise not to overemphasize conics, a greater number of illustrative problems at this place would make the student's and teacher's job somewhat easier. Moreover, no illustrative problems accompany the explanation of the hyperbola. Also a problem is needed to show the application of the rules presented for the identification of a conic.

Aside from the lack of a few illustrative problems at certain places, this text represents a distinct contribution to the needs of students and teachers of analytic geometry. The organization of the text, the style of writing, the numerous well chosen exercises, the pleasing format—all these are outstanding features. Most important of all, Dr. Middlemiss appears to have written the text for the student, whose viewpoint too frequently has been overlooked in the past. In some instances, writers—perhaps almost afraid that students might understand mathematics—have too often confused and discouraged those students who might have been not only reasonably good technicians, but also potential mathematicians of considerable merit. In short, this reviewer highly recommends the text to all teachers and students of analytic geometry.

W. B. MICHAEL

CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Michigan State College, East Lansing, Michigan.

NOTE ON LOGARITHMS AND COMPOUND INTEREST

J. S. FRAME, Michigan State College

Can the construction of a table of logarithms be explained to high school students or college freshmen without the use of integral calculus? Yes, if we admit the use of the binomial series for non-integral exponents, namely,

$$(1) \qquad (1+x)^r = 1 + rx + r(r-1)x^2/2! + r(r-1)(r-2)x^3/3! + \cdots,$$

which is valid for $|x| < 1$.

Let $y = nr$ be the simple interest on one dollar for n periods at the rate r per period, and let $A = (1+r)^n$ be the compound amount for the same period of time. Then $x = A - 1$ is the compound interest. For each choice of r we define a base number $b = (1+r)^{1/r}$, and find

$$(2) \qquad 1 + x = A = b^y, \qquad y = \log_b (1+x).$$

The quantity A^r may not only be expanded as in equation (1), but also, if $|r| < 1$, in the form

$$(3) \qquad (1+r)^y = 1 + yr + y(y-1)r^2/2! + y(y-1)(y-2)r^3/3! + \cdots.$$

Equating the expressions in (3) and (1), subtracting 1 from each side of the equality, and dividing by r, gives the following equation:

$$(4) \qquad \begin{aligned} &y + y(y-1)r/2! + y(y-1)(y-2)r^2/3! + \cdots \\ &\qquad = x + (r-1)x^2/2! + (r-1)(r-2)x^3/3! + \cdots. \end{aligned}$$

The essential idea of a limit enters at this point, when we let r approach zero on both sides of (4). We assume that both sides are still equal when $r = 0$. The left-hand side reduces to y, and the right-hand side to $x - x^2/2 + x^3/3 - \cdots$. The base $b = (1+r)^{1/r}$ approaches a limiting value $e = 2.718281828 \cdots$, which is called the natural base, and we have

$$(5) \qquad y = \log_e (1+x) = x - x^2/2 + x^3/3 - x^4/4 + x^5/5 - \cdots.$$

By studying simple and compound discount instead of simple and compound interest, formula (5) can be shown to hold for negative as well as positive values of x, provided that $|x| < 1$. Substituting in (5) the values $x = 1/5$, $-1/5$, $-1/10$, $1/125$, *etc.*, in turn, we can compute the numbers

$$(6) \qquad A = \log_e 6/5, \quad B = -\log_e 4/5, \quad C = -\log_e 9/10, \quad D = \log_e 126/125.$$

Then we use the laws of logarithms to obtain the following natural logarithms:

$$\log_e 2 = 2A + B + C, \quad \log_e 3 = 3A + 2B + C, \quad \log_e 5 = 4A + 3B + 2C, \tag{7}$$
$$\log_e 7 = 4A + 4B + 3C + D, \quad \log_e 10 = 6A + 4B + 3C.$$

Other natural logarithms of integers can be found in a similar manner, but care should be taken to compute the auxiliary logarithms A, B, C, D, *etc.*, to one more decimal place then is required in the final answer, to allow for the multiplication of errors by the coefficients in (7).

Interpolation from a known $\log_e N$ to an unknown $\log_e(N+f)$, where $N>10$ and $|f|<1$, is given accurately to six decimals by the formula

$$\log_e (N + f) = \log_e N + \frac{2f}{2N + f} + \frac{1}{12}\left(\frac{2f}{2N + f}\right)^3, \tag{8}$$

as can be shown by setting $f=xN$ and using the series (5).

To convert from natural logarithms to logarithms to the base 10, all that is required is to divide each natural logarithm, as computed above, by the value of $\log_e 10$ found from (7).

CLUB REPORTS 1944–45

Regis College Mathematics Club, Weston, Massachusetts

A formal candlelight installation ceremony for the new officers was held in September at our log cabin, Carondolet. New members received at this time the key charm, symbol of membership in the club. Evening socials were held on festive occasions throughout the year. Topics discussed at the monthly meetings included

Mathematics as a future
Air navigation
Soap bubbles
Methods of apportionment in Congress.

The mechanism of the seismograph and the tabulation of the data from this instrument were explained by Reverend Daniel Linehan, S.J., to the members of *RCMC* who visited the Weston College Seismic Station in January.

Plans were carried out at the monthly meetings for the construction of mathematical models and for a multiplex of pictures of the *RCMC* alumnae at their present occupations. The models, the multiplex, and the honor problems of the department were displayed at the close of the second semester in an exhibit sponsored by the Club. Among the models were (1) the regular solids, (2) stringed models of Desargues' configuration in a plane and in space, of a cylinder and a tangent plane convertible into a hyperboloid of one sheet with its tangent plane, of two intersecting oblique cones having their bases in the same plane and having equal altitudes, (3) knots, Peaucellier cells, and many constructions made with yarn and poster paper. Miss Marion Stark aided us in this project with many valuable suggestions.

The quarterly publication of the *RCMC*, *The Mathematical Angle*, had an increased circulation during this year. The fourth number became a summer *round robin*, in which each member wrote of her summer activities.

During the year, a representative number of members from the *RCMC* attended the biweekly meetings of the Harvard Mathematical Club.

Officers for the year were: President, Marguerite Carell; Vice-President, Mary O'Sullivan; Secretary, Annemarie Bailey; Treasurer, Jacqueline Penez. The Faculty Adviser was Sister Leonardo.

Kappa Mu Epsilon, Chicago Teachers College

The activities of the Illinois Gamma Chapter were varied, consisting of talks by faculty and students, a picnic, a party, and the installation of the Illinois Delta Chapter at St. Francis College, Joliet, Illinois. Our own chapter initiated 23 members. Among the topics discussed were:

Time—the fourth dimension, by Dr. J. T. Johnson

The oriental game of Nim, by Bernard Malina

The law of probability and combinations applied to biology, by Dr. E. C. Colin of the Biology Department

Mathematical recreations, by Professor Joseph Urbancek

Officers for the year 1944–45 were: President Archimedes, Ruth Mary Thometz; Vice-President Euler, Elaine Drews; Secretary Galileo, Veronica Stuart; Treasurer Kepler, Gertrude Koehl. Officers for 1945–46 are: President Archimedes, Patricia Powers; Vice-President Euler, Jeanne Anderson; Secretary Galileo, Alice Martinson; Treasurer Kepler, Dorothy Tisevich. During both years Professor Joseph Urbancek acted as Corresponding Secretary Descartes and Faculty Sponsor.

Mathematics Club, Immaculate Heart College, Hollywood

Monthly meetings were held from October, 1944 to May, 1945, at which papers were given by a faculty member, a guest speaker, and three student speakers. Their titles were as follows:

Differential equations, by Peggy Haws

The solar system, by Lucille Kos

Mathematics in music, by Mr. Frank Darvas of the Department of Music

Non-euclidean geometry, by Sister Margaret Ann

The future of geometry—wishful thinking, by Dr. P. H. Daus, Head of the Mathematics Department at the University of California at Los Angeles.

The officers for the year 1944–45 were: President, Zilda Cross; Vice-President, Kitty Campion; Secretary-Treasurer, Sister Margaret Ann; Program Chairman, Peggy Haws; Faculty Adviser, Peggy Masterson. Officers for 1945–46 are: President, Kitty Campion; Vice-President, Sarah Doherty; Secretary-Treasurer, Patricia Goodbody; Faculty Adviser, Dr. Myrtle Collier.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND HOWARD EVES

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to Howard Eves, College of Puget Sound, Tacoma 6, Washington.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 716. *Proposed by H. E. G. P.*

On April 1, 1946, the Erewhon Daily Howler carried the following item: "The famous astrologer and numerologist of Guayazuela, the Professor Euclide Paracelso Bombasto Umbugio, predicts the end of the world for the year 2141. His prediction is based on profound mathematical and historical investigations. Professor Umbugio computed the value of the formula

$$1492^n - 1770^n - 1863^n + 2141^n$$

for $n=0, 1, 2, 3$, and so on, up to 1945 and found that all the numbers which he so obtained in many months of laborious computation are divisible by 1946. Now, the numbers 1492, 1770, and 1863 represent memorable dates: the Discovery of the New World, the Boston Massacre, and the Gettysburg Address. What important date may 2141 be? That of the end of the world, obviously."

Deflate the Professor! That is, show with little computation that the formula proposed is divisible by 1946 for $n=0, 1, 2, 3, \cdots$.

E 717. *Proposed by Orrin Frink, Jr., State College, Pa.*

In the game called "matching chips" each of the two players selects, independently of the other, either a red or a white chip from his store of chips. If both chips are found to be of the same color, then player A keeps both chips; otherwise player B keeps both chips. Red chips are worth r cents each, and white chips are worth w cents each. Develop the theory of this game.

E 718. *Proposed by Roy Dubisch, Montana State University*

Let A be a positive integer of r digits given by $A=\sum_{i=1}^{r} x_i 10^{i-1}$ and define $D(A)=(\sum_{i=1}^{r} x_i)^2$. By $D^n(A)$, where n is a positive integer, we mean the result of applying the operator, D, n successive times to A. Prove that for every A there exists an n such that $D^n(A)=1$, 81, or 169. (*Cf.* Porges, *A Set of Eight Numbers*, this MONTHLY [1945, 379–384].)

E 719. *Proposed by George Grossman, New York City*

A man addresses n envelopes and writes n checks in payment of n bills.

(a) If the n bills are placed at random in the n envelopes, what is the probability that each bill is placed in a wrong envelope?

(b) If the n bills and the n checks are placed at random in the n envelopes, one of each in each envelope, what is the probability that in no instance are the enclosures completely correct?

(c) In (b), what is the probabiiity that each bill and each check will be in a wrong envelope?

E 720. *Proposed by Victor Thébault, Tennie, Sarthe, France*

At the vertices of an equilateral triangle three equal circles are drawn externally tangent to the circumcircle. Show that one of the three tangents to these equal circles, from any point whatever on the circumcircle, is equal to the sum of the other two.

SOLUTIONS

A Maximal Determinant

E 680 [1945, 341]. *Proposed by Gordon Pall, McGill University*

Prove that a real determinant of order 6, with elements numerically not exceeding unity, cannot have a value greater than 160.

I. *Solution by the Proposer.* It is sufficient to prove the conclusion for determinants of order 6 having all elements equal to ± 1. For, denoting the elements of the determinant by a_{ij}, if $-1<a_{ij}<1$, and the cofactor A_{ij} is positive, replace a_{ij} by $+1$; but if A_{ij} is negative, replace a_{ij} by -1; and if $A_{ij}=0$ use either value. In any case, the determinant is either increased or unaltered by such changes, and we can keep on in this way until every element is replaced by ± 1.

Now, any determinant of order 6 with all elements ± 1 is divisible by 2^5. For, on reducing in the elementary way to order 5, every element becomes $\pm 1 \cdot 1 \pm 1 \cdot 1$, which is even, and 2 factors out of each row.

Consider the Laplace expansion by means of the 20 minor determinants of order 3 in the first three rows. Each such minor determinant either has two columns proportional (and so is zero), or can be gotten by permuting rows or columns or by changing their signs from

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix} = -4, \qquad \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{vmatrix} = -4.$$

Let a, b, c, d denote distinct non-proportional columns of three elements ± 1. Then, for example, if the first three rows of the determinant have the form $aaaabc$, only four determinants $|abc|$ do not vanish, and the Laplace expansion

will not exceed $4\cdot4\cdot4=64$. Similarly, in the case of *aaabbc*, there are six non-zero minor determinants, and the expansion cannot exceed 96. The case *aaabcd* permits at most $(1+3\cdot3)16=160$. The case *aabbcc* gives 128.

There remains only the possibility *aabbcd*. Apparently we might then have $(4+2\cdot4)16=192$. This implies, however, that the last three rows are in some order *xxyyzw*, and that the complementary minor of each non-zero determinant such as *abc* is a non-zero determinant such as *xzw*. Unless *xx* is under *cd*, a non-zero determinant *abc* or *acd* or *bcd* will be multiplied by one of the zero determinants having two columns *xx*. And if *xx* is under *cd*, then *yy* is not under *cd*, and a similar remark applies. Hence an expansion equal to 192 does not exist.

An example with determinant equal to 160 is

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 \end{vmatrix}.$$

II. *Note by John Williamson, Queens College.* It was proved by Hadamard* that, if $R=(r_{ij})$ is a square matrix of order n with real elements r_{ij}, where $-1\leqq r_{ij}\leqq 1$, then the absolute value of $|R|\leqq n^{n/2}$. Further, if $|R|$ has its maximum value $n^{n/2}$, each element $r_{ij}=\pm1$ and matrix R satisfies the orthogonality conditions $\sum_{j=1}^{n} r_{ij}r_{kj}=0$, $i\neq k$. It easily follows† that, if $n>2$, then $n\equiv0 \bmod 4$. The question whether or not an Hadamard matrix, *i.e.*, a matrix R with the maximum value $n^{n/2}$ for its determinant, exists for every $n\equiv0 \bmod 4$ has not yet been answered. In 1933 Paley† discussed the known results in this connection and extended them considerably, proving, in particular, that an Hadamard matrix exists for all values of $n=4m\leqq200$ with the possible exception of the six numbers 92, 116, 156, 172, 184, 188. A list of the known values of n for which an Hadamard matrix exists is given in a paper by Williamson.‡

As remarked in this MONTHLY,§ not much is known about the maximum value of a determinant each element of which is ±1 when $n\not\equiv0 \bmod 4$, except for values of n less than eight.‖

* Jacques Hadamard, "Résolution d'une question relative aux déterminants," Bulletin des Sciences Mathématiques, (2), vol. 17, 1893, part 1, pp. 240–246.

† R. E. A. C. Paley, "On orthogonal matrices," Journal of Mathematics and Physics, Massachusetts Institute of Technology, vol. 12, 1933, pp. 311–320.

‡ John Williamson, "Hadamard's determinant theorem and the sum of four squares," Duke Mathematical Journal, vol. 11, 1944, pp. 65–81.

§ *Editorial Note* to problem E 264, this MONTHLY, [1945, 342].

‖ John Williamson, "Hadamard's determinant theorem," (abstract), this MONTHLY, [1945, 417].

Let K_n be a determinant of order n each element of which is ± 1. It is easily shown that $K_n = 2^{n-1}\Delta_{n-1}$, where Δ_{n-1} is a determinant of order $n-1$ each element of which is 1 or 0. Conversely, if Δ_{n-1} is known, the corresponding K_n may be written down, and K_n has a maximum value if and only if Δ_{n-1} has. If D_{n-1} denotes any Δ_{n-1} with a maximum value, then the problem under consideration is to show that $D_5 = 5$.

Let the matrix of D_{n-1} be $A = (a_{ij})$ and let $D_{n-1} = |A|$ be positive. Then, if $a_{ij} = 1$, its cofactor A_{ij} is positive or zero, and if $a_{ij} = 0$, then A_{ij} is negative or zero, as otherwise D_{n-1} would not have a maximum value. If

$$\Delta_n = \begin{vmatrix} q & e_1 & e_2 \cdots e_{n-1} \\ d_1 & & \\ d_2 & & A \\ d_{n-1} & & \end{vmatrix},$$

then

$$\Delta_n = q|A| - d_1\sum_{j=1}^{n} e_j A_{1j} - d_2\sum_{j=1}^{n} e_j A_{2j} - \cdots - d_{n-1}\sum_{j=1}^{n} e_j A_{n-1,j}$$

$$= q\Delta_{n-1} - \sum_{i,j=1}^{n} d_i e_j A_{ij}.$$

Since each d_i or e_j has the value 1 or 0,

$$\Delta_n = q\Delta_{n-1} - \Sigma' A_{ij}, \tag{1}$$

where $\Sigma' A_{ij}$ is the summation of the cofactors A_{ij} of all the elements of a submatrix of A. The submatrix is formed from the rows for which the corresponding d_i have the value 1 and from the columns for which the corresponding e_j have the value 1. If $q = 1$, Δ_n will have its largest absolute value when $\Sigma' A_{ij}$ has the largest negative value. If $q = 0$, Δ_n will have its largest absolute value when $\Sigma' A_{ij}$ has the largest positive value. For small values of n it is easy to determine at a glance what submatrices of A to choose to make the absolute value of Δ_n a maximum. Since it can be shown that for $n \leqq 6$ any D_n contains a first minor which is a D_{n-1}, starting with $D_1 = 1$ and applying the above process successively we arrive at two types of $D_2 = 1$,

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix},$$

one type of $D_3 = 2$,

$$\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix},$$

two types of $D_4 = 3$,

$$\begin{vmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{vmatrix},$$

and three types of $D_5=5$,

$$\begin{vmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{vmatrix}, \quad \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix}, \quad \begin{vmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{vmatrix}.$$

By applying the same process to each of the three types of D_5 the largest value of Δ_6 obtained was 9. Therefore $D_6=9$.

This last result is interesting for two reasons. First, by Hadamard's theorem, $K_7=2^6m\leqq 7^{7/2}$, and m might have been as large as 14 instead of being only 9. Second, any first minor of an Hadamard determinant H_8 gives rise to a Δ_7 of value 8. Therefore, *no D_7 contains a first minor which is a D_6.*

The following unpublished results may all be obtained by suitable applications of (1). There exist a Δ_8 of value 56, a Δ_9 of value 120, and a Δ_{10} of value 264, which fall far short of the possible maximum values 76, 195, and 521 respectively given by Hadamard's theorem. However, the value 264 for Δ_{10} is larger than the value 243 of Δ_{10} obtained from any first minor of an Hadamard determinant H_{12}. Since $D_{10}\geqq 264>243$, *no D_{11} contains a first minor which is a D_{10}.*

Finally, if an Hadamard determinant H_{4m} exists, there exists a Δ_{4m} with the value $(5m-3)m^{2m-1}$. Although no larger value for Δ_{4m} was found when $m=2$, 3, 4, the existence of a Δ_{20} with the value $25(5^9)$ shows that the value $(5m-3)m^{2m-1}$ is by no means the best for larger values of m.

Also solved by Margaret Olmsted.

Three Fractions Whose Sum is an Integer

E 682 [1945, 395]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Find three positive integers x, y, z, such that the sum of ratios

$$\frac{y}{z}+\frac{z}{x}+\frac{x}{y}$$

is an integer. (Hint: suppose $x+y+z=23$.)

Solution by Paul Erdös, Stanford University, and Ivan Niven, Purdue University. We prove that there are infinitely many solutions in integers x, y, z with $(x, y, z)=1$. The sum of the fractions equals

$$\frac{x^2z + y^2x + z^2y}{xyz}.$$

Any prime p which divides one of x, y, z clearly must divide another. For otherwise two and only two terms of the above numerator would be divisible by p. Suppose $p^\alpha \| x$ (this means that p^α divides x, but $p^{\alpha+1}$ does not) and $p^\beta \| y$, and $(p, z) = 1$. Then the fraction has the form

$$\frac{k_1p^{2\alpha} + k_2p^{\alpha+2\beta} + k_3p^\beta}{k_4p^{\alpha+\beta}},$$

where the k's are prime to p. If $2\alpha \neq \beta$ then the numerator is divisible by p to the power min $(2\alpha, \beta)$, which is less than $\alpha+\beta$. Hence $2\alpha = \beta$.

Thus if $(x, y) = e$, it follows that e^2 divides y. Consequently, denoting (y, z) and (z, x) by f and g respectively, we know that $x = eg^2$, $y = fe^2$, $z = gf^2$. The original fraction becomes

$$\frac{e^3 + f^3 + g^3}{efg}, \tag{1}$$

and we show that this is an integer for infinitely many integral values e, f, g with no common factor, in fact with $g = 1$.

For, if we have one solution of the congruence

$$e^3 + f^3 + 1 \equiv 0 \pmod{ef},$$

another solution is the pair of integers f, $(f^3+1)/e$, and this is a new solution if $e < f$. The solution suggested by the proposer is given by $e = 2$, $f = 3$, $g = 1$, whence $x = 2$, $y = 12$, $z = 9$. Our procedure, however, does not give all solutions of the congruence from this basic solution, *e.g.* $e = 5$, $f = 9$.

The above congruence is equivalent to the two simultaneous congruences $e^3 \equiv -1 \pmod{f}$ and $f^3 \equiv -1 \pmod{e}$. Congruences of this sort have been studied by Jacobsthal (*Compositio Math.*, 6 (1939), pp. 407–427).

Also solved by Murray Barbour, W. G. Brady, H. N. Carleton, P. W. A. Raine, W. J. Robinson, Joseph Rosenbaum, E. P. Starke, and the proposer. Robinson, Rosenbaum, and Starke found condition (1) above. Robinson computed the following particular primitive solutions (x, y, z): (1, 1, 1), (1, 2, 4), (2, 12, 9), (3, 18, 4), (14, 960, 25), (5, 350, 196), (3, 126, 196), (14, 588, 9), (225, 4941, 16605), (405, 1525, 33489). Rosenbaum gave the particular solutions corresponding to $e = 5$, $f = 7$, $g = 18, 78$; $e = 1$, $f = 14$, $g = 3, 5, 45, 61, 549, 915$. Starke gave the solutions corresponding to $(e, f, g) = (1, 1, 1)$, $(1, 1, 2)$, $(1, 2, 3)$, $(1, 2, 9)$, $(1, 5, 9)$, $(1, 5, 14)$, $(1, 9, 146)$, $(1, 9, 365)$, $(1, 14, 915)$, $(1, 365, 5403014)$, $(2, 3, 7)$, $(2, 5, 133)$, $(2, 7, 13)$, $(2, 7, 27)$, $(2, 7, 117)$, $(2, 13, 21)$, $(2, 13, 63)$, $(2, 13, 735)$, $(9, 19, 3794)$. Barbour, Brady, Carleton, and Raine found the solution (1, 2, 4). The proposer found the solution (2, 12, 9) and pointed out that the solution (1539, 1369634, 129549924) appeared in the *Gazeta Matematica*, Bucarest, 1931, p. 73.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4198. *Proposed by C. D. Olds, San Jose State College*

Prove that

$$\frac{1}{(n-1)!}\int_n^\infty w(t)e^{-t}dt < \left(\frac{2}{e}\right)^n,$$

where t is real, n is a positive integer, and $w(t)=(t-1)(t-2)\cdots(t-n+1)$.

4199. *Proposed by N. J. Fine, Indianapolis, Ind.*

Let r be any integer greater than unity, n a non-negative integer, and α a non-negative integer less than r. Let ν be the number of digits in the expression of n in the scale r which are not less than $r-\alpha$, and let σ be the sum of the digits. Show that

$$\sum_{k=1}^{\infty}\left[\frac{n+\alpha r^{k-1}}{r^k}\right] = \nu + \frac{n-\sigma}{r-1},$$

where the square brackets denote the greatest integer function.

4200. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Given in a plane the triangle ABC and the fixed point P which is center of a variable circle (P). Find the locus of the radical center of the circles passing through A, B, C, respectively, which have with (P) the sides BC, CA, AB as radical axes. Consider the analogous problem for a tetrahedron and a sphere with fixed center, and show that the locus is a twisted cubic through the vertices and centroid of the tetrahedron.

4201. *Proposed by Victor Thébault, Tennie, Sarthe, France*

A tetrahedron is given for which the difference of squares of opposite edges is the same for the three pairs. (1) The three medians are equal and the line joining the circumcenter to the centroid is perpendicular to one of the faces. (2) One of the altitudes passes through the symmetric of the orthocenter of the face corresponding with respect to the circumcenter of the face. (3) The sum, or the difference, of the cosines of two opposite dihedrals of the tangential tetrahedron is the same for the three pairs.

SOLUTIONS

Resultant of Nine Forces

4030 [1942, 203]. *Revised. Proposed by Victor Thébault, Tennie, Sarthe, France*

The resultant of the nine forces represented by the distances to the sides of a triangle ABC from the centers of the equilateral triangles constructed exteriorly or interiorly on its sides is represented in magnitude and direction by the vector $\overline{GG'}$ equipollent to the vector $\overline{OH'}$, where G, O, H' are the centroid of ABC, the circumcenter of ABC, and the orthocenter of the orthic triangle (that is, the triangle with vertices at the feet of the altitudes of ABC).

Generalization by R. Goormaghtigh, Bruges, Belgium. We will prove the following general theorem:

Let $BA'C$, $CB'A$, $AC'B$ be similar isosceles triangles constructed exteriorly or interiorly on the sides of a triangle ABC, the angles at A', B', C' being θ. The resultant of the nine forces represented by the distances from A', B', C' to the sides of ABC passes through the point Q dividing the segment joining the circumcenter O to the orthocenter H so that

$$OQ:QH = (1 - \cot^2 \theta/2):(1 + \cot^2 \theta/2);$$

the resultant is equipollent to the vector OH', where H' is the orthocenter of the orthic triangle.

Let A_0, B_0, C_0 be the mid-points of BC, CA, AB; M the mid-point of B_0C_0; A_1, the foot of the altitude AH. The distances from B' and C' to BC are then

$$B'B'' = 1/2AA_1 \pm \lambda A_1C, \qquad C'C'' = 1/2AA_1 \pm \lambda BA_1,$$

where

$$\lambda = 1/2 \cot \theta/2,$$

and the double signs corresponding to the cases where A', B', C' are exterior or interior to the triangle.

The distances of M to $B'B''$ and $C'C''$ are both equal to

$$1/4BC \pm \lambda AA_1.$$

Hence the resultant R_a of $A'A_0$, $B'B''$, $C'C''$ is

$$R_a = AA_1 \pm \lambda(A_1C + BA_1) \mp \lambda BC = AA_1.$$

For any value of θ, the resultant of the three considered forces perpendicular to one of the sides of the triangle ABC is equipollent to the altitude dropped from the opposite vertex.

The distance x from M to R_a is such that, $(AC>AB)$,

$$R_a x = \mp \lambda/2BC \cdot A_1A_0 + (1/2AA_1 \pm \lambda A_1C)(1/4BC \pm \lambda AA_1) - (1/2AA_1 \pm \lambda BA_1)(1/4BC \pm \lambda AA_1),$$

or

$$x = 2\lambda^2 A_1 A_0,$$

and this proves the first part of the theorem.

The resultant of the nine forces is equipollent to the resultant F of three forces equipollent to the altitudes applied to O.

Let t_1, t_2, t_3 be the complex coördinates of the vertices, the circumcircle being taken as base-circle, and denote $t_1+t_2+t_3$ by σ. Then σ is the coördinate of H and the points A_2, B_2, C_2, where AH, BH, CH cut the circumcircle again, are $-t_2t_3/t_1$, $\cdots$. Hence, h being the coördinate of the orthocenter of $A_2B_2C_2$,

$$F = \sum [(\sigma - t_2t_3/t_1)/2 - t_1] = [\sigma - \sum (t_2t_3)/t_1]/2 = (\sigma + h)/2,$$

which proves the second part of the theorem, as $A_2B_2C_2$ is similar to the orthic triangle, the center of similitude being H and the ratio 2.

Centers of Similitude

4147 [1945, 47]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Two circles varying in magnitude and position roll on two fixed circles. Find the loci of their centers of similitude. If the straight line of their centers has a constant direction, the midpoint of the segment of their centers describes a straight line.

Partial Solution by Otto Dunkel, Washington University. The two fixed circles in a plane (C_1), (C_2) have the indicated centers and the radii R_1, R_2, $R_1>R_2$, and we may suppose that the segment $C_1C_2=2c$ entends horizontally from left to right. The point P is in their plane so that $C_1P=r\pm R_1$ and $C_2P=r\pm R_2$. There are two cases such that in case I there is a subcase for which the upper signs are used and in the second subcase the lower signs are used. In the first subcase of I we have $C_1P-C_2P=R_1-R_2=2a$, and in the second $C_2P-C_1P=R_1-R_2$. Hence as r and P vary, P describes in the first subcase the right branch of a hyperbola γ with foci C_1 and C_2, center at the midpoint O of C_1C_2, and diameter of length $2a$. In the second subcase P describes the left branch of γ. There are boundary cases given by the two straight lines each tangent to C_1 and C_2 and meeting in the external center of symmetry of these two circles. If T_1 and T_2 are the points of tangency of one such common tangent then the straight line through O and the midpoint of T_1T_2 is an asymptote of γ. Let x, y be the rectangular coordinates of P with the origin at O and the positive x axis in the direction C_1C_2. Then in case I for the right branch of γ we have

$$(1) \qquad \begin{gathered} (r + R_1)^2 = (x + c)^2 + y^2; \qquad (r + R_2)^2 = (x - c)^2 + y^2; \\ 2r(R_1 - R_2) + (R_1^2 - R_2^2) = 4cx. \end{gathered}$$

Now let $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ be two points on this branch of γ and let $E_{12}(x_e, y_e)$ be the external center of symmetry of (P_1), (P_2), circles with the indicated centers and radii r_1, r_2. Then $(r_2-r_1)x_e=r_2x_1-r_1x_2$ and the equation of the radical axis of (C_1) and (C_2) is $4cx-(R_1^2-R_2^2)=0$; and the last equation in (1) gives two

equations corresponding to P_1 and P_2. The elimination of the first term from these two equations gives

$$(2) \qquad \begin{aligned} (r_2 - r_1)(R_1^2 - R_2^2) &= 4c(r_2x_1 - r_1x_2) = 4c(r_2 - r_1)x_e, \\ 4cx_e &= (R_1^2 - R_2^2), \qquad r_2 \neq r_1, \end{aligned}$$

and hence E_{12} lies on the radical axis of the two fixed circles (C_1) and (C_2). If $r_2=r_1$ and $x_2 \neq x_1$, it is obvious that this conclusion is again true. For the left branch we have merely to change the sign of both R_1 and R_2 in the above, and hence the final result is the same.

In case II there are also two subcases; in the first $C_1P=r+R_1$, $C_2P=r-R_2$, and the second subcase is obtained by altering the sign of both R_1 and R_2. It will suffice to consider the first. The last equation in (1) gives the corresponding equation needed here by changing the sign of R_2 only, and we then have

$$(1)' \qquad 2r(R_1 + R_2) + (R_1^2 - R_2^2) = 4cx.$$

In case II we have a hyperbola γ' if $R_1+R_2=2a<2c=C_1C_2$, and the first subcase gives the right branch while the other subcase gives left. Since the next step eliminates the first term of (1)′ the equations corresponding to (2) will be the same. Thus, finally, in both cases I and II the external center of symmetry of (P_1) and (P_2) lies on the radical axis of the fixed circles (C_1) and (C_2).

For a complete discussion the internal centers of symmetry of (P_1) and (P_2) should be considered; but in this study no results of interest have been obtained.

The remaining part of the problem is obvious from elementary theorems. For, if there are several pairs of points P_1 and P_2 such that the lines P_1P_2 are parallel, then since P_1 and P_2 lie on γ, or γ', the midpoints of the chords P_1P_2 lie on a diameter of the corresponding hyperbola.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

Associate Professor Garrett Birkhoff of Harvard University has been elected as the Academico Correspondiente of the Academia Nacional de Ciencias Exactas Fisicas y Naturales de Lima and Socio Honorario of the Mexican Mathematical Society.

At the University of California, Assistant Professor A. P. Morse has been promoted to an associate professorship and Associate Professor C. B. Morrey to a professorship. Professor J. H. McDonald has retired.

At the University of Illinois, Drs. M. M. Day, C. W. Mendel, J. W. Peters, L. L. Steimley, and H. E. Vaughan have been promoted to assistant professorships; Assistant Professor H. J. Miles has been promoted to an associate professorship.

Dr. P. H. Anderson has been appointed Economic Analyst with the War Assets Corporation in Washington, D. C.

Dr. R. H. Bing of the University of Texas has been promoted to an assistant professorship.

Dr. W. B. Caton of Duluth State Teachers College has been promoted to an assistant professorship.

Assistant Professor Claude Chevalley of Princeton University has been promoted to an associate professorship.

Assistant Professor Helen E. Clarkson of Hardin-Simmons University has been appointed to an adjunct professorship at the University of South Carolina.

Assistant Professor J. L. Dorroh of Louisiana State University has been promoted to an associate professorship.

Dr. Benjamin Epstein has been appointed mathematician of the staff of the Cost Research Laboratory of the Carnegie Institute of Technology.

Dr. Rufus Isaacs has been appointed to an assistant professorship at the University of Notre Dame.

Dr. W. J. Kirkham has returned to Oregon State College after service in the United States Naval Reserve.

Dr. Szolem Mandelbrojt has been appointed to a professorship at the Rice Institute.

Dr. E. W. Montroll of the Polytechnic Institute of Brooklyn has been appointed to an assistant professorship at the University of Pittsburgh.

Assistant Professor Hermance Mullemeister of the University of Washington has been promoted to an associate professorship.

Dr. E. A. Nordhaus has been reappointed to an assistant professorship at Michigan State College.

G. P. Oakland of the University of Saskatchewan has been appointed Assistant Professor of Actuarial Science at the University of Manitoba.

Dr. I. E. Perlin has been appointed to an assistant professorship at the Georgia School of Technology.

Associate Professor F. A. Rickey of Louisiana State University has been promoted to a professorship.

Associate Professor L. V. Robinson of Emory University has been appointed to an associate professorship at the University of South Carolina.

Dr. P. C. Rosenbloom of Brown University has been promoted to an assistant professorship.

Assistant Professor I. J. Schoenberg of the University of Pennsylvania has been promoted to an associate professorship.

Jack Silber has returned to civilian life and is doing graduate work at the University of Chicago.

Helen F. Smith of Iowa State College of Agriculture and Mechanic Arts has been promoted to an assistant professorship.

Dr. F. E. Ulrich of the Rice Institute has been promoted to an assistant professorship.

Professor J. H. M. Wedderburn of Princeton University has retired with the title of professor emeritus.

Assistant Professor P. A. White of Louisiana State University has been appointed to an assistant professorship at the University of Southern California.

Dr. P. M. Whitman has been appointed to an assistant professorship at Tufts College.

Assistant Professor J. W. T. Youngs of Purdue University has been appointed to an associate professorship at Indiana University.

Professor Antoni Zygmund of Mount Holyoke College has been apointed to a professorship at the University of Pennsylvania.

The following appointments to instructorships are announced:
Brown University: Dr. Herbert Federer
College of the City of New York: Dr. S. F. Barber
Columbia University: Dr. E. R. Kolchin
Iowa State College: O. C. Carpenter, R. E. Carr, A. E. Engelbrecht, C. J. Maloney, R. M. Robinson
Ohio State University: Dr. V. F. Cowling
Purdue University: Dr. B. H. Arnold, Dr. W. F. Eberlin, Dr. A. L. Whiteman
The University of Chicago: A. R. Jacoby
The University of California: Dr. Abraham Seidenberg
The University of Illinois: Dr. P. W. Carruth, Dr. M. E. Munroe
The University of Michigan: Dr. L. V. Toralballa
Yale University: Dr. R. E. Fullerton, L. R. Norwood, Dr. B. J. Pettis

Professor Harry Bateman of California Institute of Technology died January 21, 1946. He was a charter member of the Association.

A. A. Blumberg of the Agricultural and Mechanical College of Texas died October 21, 1945.

Professor Emeritus A. L. Candy of the University of Nebraska died July 18, 1945.

Professor Emeritus F. H. Clutz of Gettysburg College died December 30, 1945.

Professor Emeritus A. R. Congdon of the University of Nebraska died November 11, 1945.

Professor C. W. Emmons of Simpson College, Indianola, Iowa, died December 29, 1945. He was a charter member of the Association.

Dr. Hans Fried of Swarthmore College died December 23, 1945.

Professor C. F. Gummer of Queen's University, Kingston, Canada, died January 21, 1946. He had been a charter member of the Association.

SUMMER COURSES

The following institutions announce courses in mathematics for the summer of 1946:

Columbia University. From July 8 to August 16 the following graduate courses will be offered: By Professor Artin: theory of rings, introduction to algebra. By Professor Kasner: survey of modern mathematics, differential geometry. By Professor Koopman: theory of functions of a complex variable. By Professor Lorch: the calculus of finite differences.

Ohio State University. From June 18 to August 30 the following advanced courses will be offered: By Professor Bamforth: vector analysis. By Mr. Barlas: advanced calculus. By Professor Blumberg: projective geometry, Fourier series and spherical harmonics. By Professor Mann: theory of equations.

The State University of Iowa. From June 12 to August 8 the following graduate courses will be offered: By Professor Chittenden: theory of determinants. By Professor Knowler: mathematics of business and industry. By Professor Oberg: integral equations.

The University of Chicago. From June 25 to August 31 the following graduate courses will be offered. By Professor Albert: introduction to modern higher algebra, division algebras. By Professor Barnard: linear functional operators. By Professor Hartung: advanced problems in teaching mathematics in secondary school and junior college. By Dr. Kaplansky: lattice theory. By Professor Kelley: theory of functions of a complex variable. By Professor Logsdon: analytic projective geometry. By Professor Schilling: continuous groups.

The University of Michigan. From July 1 to August 23 the following courses will be offered in addition to the standard courses in differential equations, theory of equations, solid analytic geometry and advanced calculus: By Professor Bartels: hydrodynamics. By Professor Churchill: Fourier series, operational mathematics. By Professor Craig: quality control, advanced statistics. By Professor Dwyer: statistics. By Professor Eilenberg: axiomatic set theory topology. By Professor Fischer: finite differences, elementary life insurance. By Professor Hay: elasticity. By Professor Hildebrandt: functions of a complex variable, theory of differential equations. By Professor Karpinski: teaching of algebra, history of geometry. By Professor Nesbitt: theory of matrices, finite differences. By Professor Rothe: topics in mathematical physics. By Professor Thrall: finite groups, algebraic geometry. By Professor Wilder: foundations of mathematics, studies in foundations.

The University of Minnesota. From June 19 to June 27 the following advanced courses will be given: By Professor Cameron: theory of probability, tutorial course in advanced mathematics. By Professor Carlson: modern analytic geometry. By Professor Wegner: differential equations. From July 29 to August 31: By Professor Campaigne: introduction to modern algebra, tutorial course in advanced mathematics.

The University of Pennsylvania. From July 2 to August 23 advanced courses will be given in modern analytic geometry, differential equations, introduction to modern higher algebra, and the calculus of variations.

The University of Wyoming. From June 20 to July 24 the following advanced courses will be given: By Professor Barr: differential equations. By Professor Neubauer: advanced college algebra. By Professor Rechard: differential equations. From July 25 to August 23: By Professor Alden: advanced college algebra. By Professor Barr: differential equations.

GENERAL INFORMATION

EDITED BY C. V. NEWSOM

Send information of especial interest to mathematicians, exclusive of personal items, to C. V. Newsom, Oberlin College, Oberlin, Ohio.

RELEASE OF FACULTY PERSONNEL FROM THE SERVICES

In view of the critical need for university and college faculty personnel in many institutions, the Office of War Mobilization and Reconversion has arranged with the services to expedite the release of personnel desired by the higher institutions in the United States. Dr. M. H. Trytten, Director of the Office of Scientific Personnel, has announced the following criteria and procedures for the release of faculty personnel from the services.

A. PURPOSE.

The shortage of college and university teachers is so acute that many veterans may be denied the right to educational benefits provided under the Servicemen's Readjustment Act unless steps are taken to increase the supply of competent teachers.

To aid in relieving this shortage, the War and Navy Departments have from time to time released teachers who did not have enough points for discharge. The American Council on Education has heretofore served as the clearing house for these released in the manner described in emergency supplements (dated August 27 and September 6, 1945) to *Higher Education and National Defense.*

In order to regularize and extend this practice, the Office of War Mobilization and Reconversion has created an Interagency Committee on Manpower Shortages, one of whose duties is to establish criteria, standards, and procedures for clearing requests for release from military duties. The college portion of this program will be administered by the U. S. Office of Education. No new Army and Navy policies on release are involved.

B. STANDARDS AND CRITERIA.

1. Any duly accredited institution of higher education may request the Office of Education to certify for release any member of the Armed Forces who is qualified for college teaching and the auxiliary duties related to teaching.

2. It is not necessary for the man to have been a member of the faculty of the college requesting his release.

3. Only men on duty within the continental United States may be requested from the Army. (This does not preclude a college from making representation direct to the War Department for the release of personnel on duty outside the continental United States. Men may be released who are deemed "essential to the national health, safety, or interest.") This restriction does not apply to men who are in the Navy.

4. The man whose release from the Army is sought must make his own request through military channels in the manner described in the next section of this statement. The individual whose release is sought should be warned that unless he has had overseas service he will lose mustering-out pay. These conditions are not applicable to Navy personnel.

5. The man whose release is requested must be employed full time. He may engage in teaching and the auxiliary duties related to teaching. His case will be strengthened if the college can show that his return to the campus will directly or indirectly enrich the educational services provided for veterans.

6. The college must show that the man whose release is requested is needed and that the college has not been able to secure a suitable individual from civilian sources.

7. The request for release should include a comprehensive factual statement of the duties to be performed by the individual, and an outline of his qualifications.

C. PROCEDURES.

The President or his duly constituted representative should address the request for release of military personnel that is needed for college staff members *in duplicate* to Dr. Ernest V. Hollis, Division of Higher Education, U. S. Office of Education, Washington 25, D. C. The information to be supplied differs in some particulars for Army and Navy personnel. This is due to the fact that Army policy requires the man to initiate his own request for release when it is determined that he is eligible. In turn, by law a person who requests his own release from military duty loses mustering-out pay unless he has served outside the continental United States. By Navy procedure the man who is eligible does not request his own release and hence does not take all of the steps required of soldiers. The Marine Corps follows Navy policy.

1. As a preliminary to writing the Office of Education requesting the certification of a man for release, the college should furnish the following information which the man himself usually must supply:

(a) A statement that an individual from the Army is willing to request his own release through military channels; that Navy personnel is willing to be released.

(b) Name, rank, and serial number.

(c) Branch of military service, length of service, and nature of present duty.

(d) Current Armed Forces address.

(e) If the individual is in the Army, the name and address of his commanding officer.

The letter to the Office of Education containing the above information should also present factual statement of the duties to be performed, and the date on which this service will begin, and an outline of his qualifications.

2. A copy of the statement which is sent in duplicate to the Office of Education should be furnished to the man whose release is sought. If the man is in the Army he must attach a signed copy to the request he initiates. Army personnel should not initiate this request until the college advises him of his elgibility. Navy personnel does not take this step.

3. The Office of Education will review the information submitted by a duly constituted college official for its conformity to the criteria and standards for release which have been established by the Interagency Committee of OWMR.

4. The Office of Education will certify to individuals who meet standards for release to (a) the commanding officer of Army personnel, and (b) to the Bureau of Naval Personnel for Navy and Marine Corps personnel. Copies of the certification will go to the individual, to the college, and for Army personnel to the War Department.

5. Send separate request for the release of individuals even though the release of more than one man may be requested under the same cover.

6. Any Government agency or private association which receives from a university or an individual a request for release under the conditions set forth in this circular should forward the information to the U. S. Office of Education or advise the inquirer of the approved procedures for securing consideration.

7. Certification of an individual by the Office of Education for release does not guarantee that the Army or the Navy will release the individual in question; the Armed Services necessarily reserve the right to retain men whom they consider essential for military purposes.

MATHEMATICS REQUIRED FOR GRADUATION IN KANSAS

For about twenty years there has been no state requirement in Kansas that high school graduates must have studied some mathematics. So on April 15, 1944, the Kansas Association of Teachers of Mathematics and the Kansas Section of the M.A.A. joined in requesting the State Department of Education to establish a requirement of at least one unit of mathematics for graduation from accredited high schools of Kansas. This action was taken after the adoption of the recommendation which follows.

"The K. A. T. M. Committee on the Improvement of Instruction recommends that the Kansas Association of Teachers of Mathematics and the Kansas Section of the Mathematical Association of America unite in requesting the Kansas State Department of Education to require a minimum of one unit of mathematics for graduation from accredited high schools of the state. The Committee recommends that the secretaries of the two organizations submit this request to the State Department and that their communication include the following explanations and suggestions.

"This request is made because the members of these organizations believe that mathematics study is of fundamental importance in the general education of every individual. This request does not imply that every high school pupil should study algebra. All high schools should provide, for boys and girls who are unable to profit from the study of algebra, a course which is based on principles and applications of arithmetic which nearly everyone can use to advantage in everyday living. It is urged, however, that superior students should not be enrolled in this course. Students who have sufficient ability and a good understanding of arithmetical principles when they enter high school should be given an opportunity to extend their knowledge of mathematics through the study of algebra.

"The administrators of some small high schools may feel that they cannot have two ninth grade courses in their schools and that if this requirement were made they would be forced to choose between the basic arithmetic course and elementary algebra. Even in the smallest schools there are two classes of ninth grade mathematics. One could be arithmetic and the other algebra. In very small schools these two courses could be offered in alternate years.

"Perhaps one-third to one-half of entering ninth grade pupils should be enrolled in the arithmetic course, and the remainder in the algebra course in the ninth year. The placement of pupils in these two courses should be on the basis of general ability and knowledge of fundamental arithmetic, rather than on any plans which the students may have regarding future college attendance. There are at least two reasons for this. First, boys and girls cannot know for certain whether they will go to college. Second, if the high school is to be the student's last opportunity at formal education, it is all the more important that he be given an opportunity there to gain some basic understanding of mathematical methods that are used by persons in many important fields.

"At the end of the ninth year, it is probable that some of the pupils who have done at least reasonably good work in the arithmetic course should continue with mathematics study, and they should be enrolled in algebra the tenth year. Some of them should continue on through plane geometry and perhaps even another year of mathematics.

"Every school, regardless of size, should offer a year of plane geometry and either require or encourage enrollment in this course of all students who are capable of pursuing it with understanding. It is highly desirable that at least one additional year of mathematics be offered. In very small schools, the plan of offering courses in alternate years can be used to make available more advanced courses.

"This is not a wartime proposal. Although the needs of the armed forces and war industries have focused the attention of the schools and the public on deficiencies in the mathematics instruction of our youth, the need for sound basic training in mathematics will always be present as long as we are concerned with educating boys and girls to understand their environment and to become useful citizens."

As a result of this united effort on the part of the mathematical organizations of Kansas, the Kansas High School Handbook for 1945 lists a minimum of one unit of mathematics among the requirements for high school graduation. At the time of the negotiations with the State Department of Education, the mathematical organizations also sent a letter containing some suggestions for the high school mathematics program to all the high school principals in the state.

MATHEMATICAL ASSOCIATION OF AMERICA

REPORT OF THE TREASURER FOR THE YEAR 1945

The following report of the Secretary-Treasurer as Treasurer for the year 1945 has been approved by the Finance Committee and accepted by vote of the Board of Governors.

I. Total Funds of the Association on December 31, 1944

(See Treasurer's report, pp. 230–233 of the Monthly for April, 1945)

Current Fund (checking account)		$ 5,474.49
Savings Account (Ithaca Savings Bank)		1,000.00
Savings Account (Oberlin Savings Bank)		648.96
Invested Funds (Cleveland Trust Company)		
Carus Fund	$ 8,353.37	
Chace Fund	9,347.93	
Houck Fund	8,676.14	
Chauvenet Fund	614.04	
Life Membership Fund	755.03	
General Fund	21,916.26	49,662.77
		$56,786.22

II. Current Fund Account for 1945

Receipts		Expenditures	
Balance, Jan. 1, 1945	$ 5,474.49	Monthly	
Dues	9,001.16	Publication	$ 5,094.19
Initiation fees	248.00	Reprints	70.79
Subscriptions	2,744.37	Editor-in-Chief's office	422.36
Sale of back numbers Monthly	540.91	Secretary-Treasurer's office	
Advertisements	543.50	Clerical help	3,268.95
Sale of Archibald's Outline	156.75	Postage	387.66
Interest on General Fund	791.66	Printing	264.29
Interest on Carus Fund	301.58	Office supplies	243.14
Interest on Chace Fund	337.49	Bank charges	108.75
Interest on Houck Fund	314.15	Exec. and Finance Committees	102.38
Interest on Chauvenet Fund	23.34	Regional Governors	184.51
Interest on Life Membership Fund	26.93	Sections	86.39
Interest from Hardy Fund	120.00	Subventions	
Sale of monographs (Carus)	523.35	Amer. Math. Society	100.00
Sale of Papyrus (Chace)	137.95	Mathematical Reviews	350.00
Miscellaneous sources	4.14	National Math. Magazine	200.00
Transferred from General Fund	1,264.79	Math. Cuneiform Texts (Chace)	400.00
		Back numbers Monthly	118.15
		Meetings	167.33
		Coordinating Committee	122.64
		Coop. Committee on Sci. Teaching	39.78
		To General Fund	6,078.70
		Subscriptions to Annals	12.50
		B. F. Finkel (Hardy Fund)	120.00
		Transferred to Carus Fund	824.93
		Transferred to Chace Fund	75.44
		Transferred to Houck Fund	314.15
		Transferred to Chauvenet Fund	23.34
		Transferred to Life Memb. Fund	26.93
		Balance, Dec. 31, 1945	3,347.26
	$22,554.56		$22,554.56

III. Savings Account, Ithaca Savings Bank

Balance, Jan. 1, 1945	$1,000.00		
Interest	7.50	Balance, Dec. 31, 1945	$1,007.50
	$1,007.50		$1,007.50

IV. Invested Funds, Cleveland Trust Company

Cash Balance, Jan. 1, 1945	$ 40.77	Market value of securities, Dec. 31, 1945	$57,113.00
Market value of securities, Dec. 31, 1944	49,622.00	Cash balance, Dec. 31, 1945	279.92
From Current Fund	6,078.70		
Increase in value of securities	1,651.45		
	$57,392.92		$57,392.92

List of Securities

	Par Value	Market Value Dec. 31, 1945
U. S. Savings Bonds, 1947	$ 700.00	$ 658.00
U. S. Treasury Bonds, 2%, 1950	3,000.00	3,060.00
U. S. Treasury Bonds, 2%, 1954	2,000.00	2,080.00
U. S. Treasury Bonds, 2½%, 1969	2,000.00	2,060.00
U. S. Treasury Bonds, 2½%, 1972	1,000.00	1,010.00
U. S. Treasury Bonds, 2½%, 1952	5,000.00	5,050.00
U. S. Treasury Bonds, 1½%, 1948	2,000.00	2,040.00
U. S. Savings Bonds, Ser. G, 2½%, 1953	3,000.00	2,850.00
U. S. Savings Bonds, Ser. G, 2½%, 1954	8,200.00	7,790.00
Canadian Nat. Ry. Co. Bonds, 4½%, 1956	2,000.00	2,420.00
Gatineau Power Co. 1st Mort. Bonds, Ser. A, 4½%, 1970	2,000.00	2,120.00
C. and O. Ry. Co. Ref. Bonds, Ser. D, 3½%, 1996	3,000.00	3,180.00
Union Pacific R.R. Co. Deb. Bonds, 3½%, 1971	3,000.00	3,210.00
Amer. Tel. & Tel. Co. Conv. Deb. Bonds, 3%, 1956	2,000.00	3,020.00
Columbus and So. Ohio Elec. Co. Bonds, 3¼%, 1970	2,000.00	2,200.00
Commonwealth Edison Co. Conv. Deb. Bonds, 3½%, 1958	2,000.00	2,680.00
New York Steam Corp. 1st Mort. Bond, 3½%, 1963	1,000.00	1,060.00
Amer. Tobacco Co. Bonds, 3%, 1969	4,000.00	4,240.00
C. and O. Ry. Co. common stock, 25 shares		1,400.00
Amer. Tel. & Tel. Co. common stock, 10 shares		1,910.00
Standard Oil Co. New Jersey common stock, 20 shares		1,320.00
Atch. Top., Santa Fe R.R. non cum. pfd. stock, 15 sh.		1,755.00
		$57,113.00

V. Carus Fund

Balance, Jan. 1, 1945	$8,353.37		
Sale of monographs	523.35		
Interest	301.58		
Increase in value of securities	277.45	Balance, Dec. 31, 1945	$9,455.75
	$9,455.75		$9,455.75

VI. Chace Fund

Balance, Jan. 1, 1945	$ 9,347.93	Mathematical Cuneiform Texts	$ 400.00
Sale of Papyrus	137.95		
Interest	337.49		
Increase in value of securities	310.47	Balance, Dec. 31, 1945	$ 9,733.84
	$10,133.84		$10,133.84

VII. Houck Fund

Balance, Jan. 1, 1945	$8,676.14		
Interest	314.15		
Increase in value of securities	289.00	Balance, Dec. 31, 1945	$9,279.29
	$9,279.29		$9,279.29

VIII. Chauvenet Fund

Balance, Jan. 1, 1945	$ 614.04		
Interest	23.34		
Increase in value of securities	21.47	Balance, Dec. 31, 1945	$ 658.85
	$ 658.85		$ 658.85

IX. Life Membership Fund

Balance, Jan. 1, 1945	$ 755.03	To General Fund	$ 88.64
Interest	26.93		
Increase in value of securities	24.77	Balance, Dec. 31, 1945	718.09
	$ 806.73		$ 806.73

X. General Fund

Balance, Jan. 1, 1945	$21,916.26	Transferred to Current Fund	$ 1,264.79
From Current Fund	6,078.70		
Increase in value of securities	728.29		
From Life Membership Fund	88.64	Balance, Dec. 31, 1945	27,547.10
	$28,811.89		$28,811.89

XI. Total Funds of the Association, December 31, 1945

Current Fund (checking account)		$ 3,347.26
Savings Account (Ithaca Savings Bank)		1,007.50
Savings Account (Oberlin Savings Bank)		648.96
Invested Funds (Cleveland Trust Company)		
Carus Fund	$ 9,455.75	
Chace Fund	9,733.84	
Houck Fund	9,279.29	
Chauvenet Fund	658.85	
Life Membership Fund	718.09	
General Fund	27,547.10	57,392.92
		$62,396.64

THE NEW EDITOR-IN-CHIEF

By recent action of the Board of Governors Professor C. V. Newsom of Oberlin College was elected Editor-in-Chief of the Monthly for a five year period beginning January 1, 1947. Professor Newsom has been an Associate Editor since 1943. During this time he has conducted the interesting and valuable department known first as War Information and later, with the coming of peace, as General Information. He is a member of the Board of Governors and of various committees. He brings to his new tasks a rich experience in the affairs of the Association.

CALENDAR OF FUTURE MEETINGS

Twenty-eighth Summer Meeting, Ithaca, New York, August 19–20, 1946.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

Allegheny Mountain, Pittsburgh, Pa., October 1946
Illinois, Normal, May 10–11, 1946
Indiana
Iowa
Kansas
Kentucky, Louisville, April 27, 1946
Louisiana-Mississippi
Maryland-District of Columbia-Virginia
Metropolitan New York, New York, N. Y., May 4, 1946
Michigan
Minnesota
Missouri
Nebraska, Omaha, May 4, 1946
Northern California, San Francisco, January 25, 1947
Ohio
Oklahoma
Pacific Northwest
Philadelphia, Philadelphia, November 30, 1946
Rocky Mountain
Southeastern
Southern California, Claremont, March 8, 1947
Southwestern
Texas
Upper New York State
Wisconsin, Milwaukee, May 4, 1946

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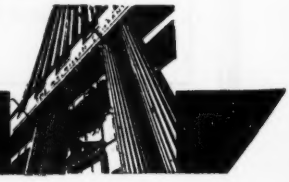

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